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PREFACE.

THIS Book is published at the request of many Teachers. It is intended to assist, *first,* Masters who cannot devote much time to the examination of the work of their pupils, and *secondly,* Students who are unable to obtain the help of competent instructors. In working the Exercises, the simplest and most obvious methods of solution have in all cases been adopted.

J. HAMBLIN SMITH.

CAMBRIDGE, *May* 1874.

ELEMENTARY ALGEBRA.

KEY.

IV.

1. $a+b+3a-2b=4a-b.$

2. $a+b-a+3b=4b.$

3. $3a+5b-6c-2a-4b+2c=a+b-4c.$

4. $a+b-c-a+b+c=2b.$

5. $14x-5x+9-\{4-3x-2x+3\}=14x-5x+9-4+3x+2x-3=14x+2.$

6. $4x-\{3x-(2x-x+a)\}=4x-\{3x-2x+x-a\}=4x-3x+2x-x+a$
$=2x+a.$

7. $15x-\{7x+(3x+a-x)\}=15x-\{7x+3x+a-x\}$
$=15x-7x-3x-a+x=6x-a.$

8. $a-[b+\{a-b-a\}]=a-[b+a-b-a]=a-b-a+b+a=a.$

9. $6a+[4a-\{8b-2a-4b-22b\}-7b]-[7b+\{8a-3b-4a+8b\}+6a]$
$=6a+[4a-8b+2a+4b+22b-7b]-[7b+8a-3b-4a+8b+6a]$
$=6a+4a-8b+2a+4b+22b-7b-7b-8a+3b+4a-8b-6a$
$=2a-b.$

10. $b-[b-a-b-\{b-(b-a+b)\}]=b-[b-a-b-\{b-b+a-b\}]$
$=b-[b-a-b-b+b-a+b]=b-b+a+b+b-b+a-b=2a.$

11. $2c-6a+b-\{c-5a-2b-a+3b\}=2c-6a+b-c+5a+2b+a-3b=c.$

12. $2x-\{a-(2a-[3a-(4a-[5a-6a+x])])\}$
$=2x-\{a-(2a-[3a-(4a-5a+6a-x)])\}$
$=2x-\{a-(2a-[3a-4a+5a-6a+x])\}$
$=2x-\{a-(2a-3a+4a-5a+6a-x)]\}$
$=2x-\{a-2a+3a-4a+5a-6a+x\}$
$=2x-a+2a-3a+4a-5a+6a-x=x+3a.$

13. $25a-19b-[3b-\{4a-5b+6c\}]=25a-19b-[3b-4a+5b-6c]$
$=25a-19b-3b+4a-5b+6c=29a-27b+6c.$

VIII.

1.
$$\begin{array}{l} x+3 \\ x+9 \\ \hline x^2+3x \\ \quad +9x+27 \\ \hline x^2+12x+27 \end{array}$$

2.
$$\begin{array}{l} x+15 \\ x-7 \\ \hline x^2+15x \\ \quad -7x-105 \\ \hline x^2+8x-105 \end{array}$$

3.
$$\begin{array}{l} x-12 \\ x+10 \\ \hline x^2-12x \\ \quad +10x-120 \\ \hline x^2-2x-120 \end{array}$$

4.
$$\begin{array}{l} x-8 \\ x-7 \\ \hline x^2-8x \\ \quad -7x+56 \\ \hline x^2-15x+56 \end{array}$$

5.
$$\begin{array}{l} a-3 \\ a-5 \\ \hline a^2-3a \\ \quad -5a+15 \\ \hline a^2-8a+15 \end{array}$$

6.
$$\begin{array}{l} y-6 \\ y+13 \\ \hline y^2-6y \\ \quad +13y-78 \\ \hline y^2+7y-78 \end{array}$$

7.
$$\begin{array}{l} x^2-4 \\ x^2+5 \\ \hline x^4-4x^2 \\ \quad +5x^2-20 \\ \hline x^4+x^2-20 \end{array}$$

8.
$$\begin{array}{l} x^2-6x+9 \\ x^2-6x+5 \\ \hline x^4-6x^3+9x^2 \\ \quad -6x^3+36x^2-54x \\ \qquad\qquad +5x^2-30x+45 \\ \hline x^4-12x^3+50x^2-84x+45 \end{array}$$

9.
$$\begin{array}{l}
x^2+5x-3\\
x^2-5x-3\\
\hline
x^4+5x^3-3x^2\\
\qquad -5x^3-25x^2+15x\\
\qquad\qquad -3x^2-15x+9\\
\hline
x^4 \qquad -31x^2 \qquad +9
\end{array}$$

10.
$$\begin{array}{l}
a^3-3a+2\\
a^3-3a^2+2\\
\hline
a^6-3a^4+2a^3\\
\qquad -3a^5+9a^3-6a^2\\
\qquad\qquad +2a^3-6a+4\\
\hline
a^6-3a^5-3a^4+13a^3-6a^2-6a+4
\end{array}$$

11.
$$\begin{array}{l}
x^2-x+1\\
x^2+x-1\\
\hline
x^4-x^3+x^2\\
\qquad +x^3-x^2+x\\
\qquad\qquad -x^2+x-1\\
\hline
x^4 \qquad -x^2+2x-1
\end{array}$$

12.
$$\begin{array}{l}
x^2+xy+y^2\\
x^2-xy+y^2\\
\hline
x^4+x^3y+x^2y^2\\
\qquad -x^3y-x^2y^2-xy^3\\
\qquad\qquad +x^2y^2+xy^3+y^4\\
\hline
x^4 \qquad +x^2y^2 \qquad +y^4
\end{array}$$

13.
$$\begin{array}{l}
x^2+xy+y^2\\
x-y\\
\hline
x^3+x^2y+xy^2\\
\qquad -x^2y-xy^2-y^3\\
\hline
x^3 \qquad\qquad -y^3
\end{array}$$

14.
$$\begin{array}{l}
a^2-x^2\\
a^4+a^2x^2+x^4\\
\hline
a^6-a^4x^2\\
\qquad +a^4x^2-a^2x^4\\
\qquad\qquad +a^2x^4-x^6\\
\hline
a^6 \qquad\qquad -x^6
\end{array}$$

15.
$$\begin{array}{l}
x^3-3x^2+3x-1\\
x^2+3x+1\\
\hline
x^5-3x^4+3x^3-x^2\\
\qquad +3x^4-9x^3+9x^2-3x\\
\qquad\qquad +x^3-3x^2+3x-1\\
\hline
x^5 \qquad -5x^3+5x^2 \qquad -1
\end{array}$$

16.
$$\begin{array}{l}
x^3+3x^2y+9xy^2+27y^3\\
x-3y\\
\hline
x^4+3x^3y+9x^2y^2+27xy^3\\
\qquad -3x^3y-9x^2y^2-27xy^3-81y^4\\
\hline
x^4 \qquad\qquad\qquad -81y^4
\end{array}$$

17.
$$\begin{array}{l}
a^3+2a^2b+4ab^2+8b^3\\
a-2b\\
\hline
a^4+2a^3b+4a^2b^2+8ab^3\\
\qquad -2a^3b-4a^2b^2-8ab^3-16b^4\\
\hline
a^4 \qquad\qquad\qquad -16b^4
\end{array}$$

18.
$$\begin{array}{l}
8a^3+4a^2b+2ab^2+b^3\\
2a-b\\
\hline
16a^4+8a^3b+4a^2b^2+2ab^3\\
\qquad -8a^3b-4a^2b^2-2ab^3-b^4\\
\hline
16a^4 \qquad\qquad\qquad -b^4
\end{array}$$

19.
$$\begin{array}{l}
a^3-2a^2b+3ab^2+4b^3\\
a^2-2ab-3b^2\\
\hline
a^5-2a^4b+3a^3b^2+4a^2b^3\\
\qquad -2a^4b+4a^3b^2-6a^2b^3-8ab^4\\
\qquad\qquad\qquad -3a^3b^2+6a^2b^3-9ab^4-12b^5\\
\hline
a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5
\end{array}$$

20.
$$\begin{array}{l}
a^3+3a^2b-2ab^2+3b^3\\
a^2+2ab-3b^2\\
\hline
a^5+3a^4b-2a^3b^2+3a^2b^3\\
\qquad +2a^4b+6a^3b^2-4a^2b^3+6ab^4\\
\qquad\qquad\qquad -3a^3b^2-9a^2b^3+6ab^4-9b^5\\
\hline
a^5+5a^4b+a^3b^2-10a^2b^3+12ab^4-9b^5
\end{array}$$

21.
$$\begin{array}{l}
a^2-2ax+4x^2\\
a^2+2ax+4x^2\\
\hline
a^4-2a^3x+4a^2x^2\\
\qquad +2a^3x-4a^2x^2+8ax^3\\
\qquad\qquad\qquad +4a^2x^2-8ax^3+16x^4\\
\hline
a^4 \qquad\qquad +4a^2x^2 \qquad\qquad +16x^4
\end{array}$$

22.
$$\begin{array}{l}
9a^2+3ax+x^2\\
9a^2-3ax+x^2\\
\hline
81a^4+27a^3x+9a^2x^2\\
\qquad\qquad -27a^3x-9a^2x^2-3ax^3\\
\qquad\qquad\qquad\qquad +9a^2x^2+3ax^3+x^4\\
\hline
81a^4 \qquad\qquad +9a^2x^2 \qquad\qquad +x^4
\end{array}$$

23.
$$\begin{array}{l}
x^4-2ax^2+4a^2\\
x^4+2ax^2+4a^2\\
\hline
x^8-2ax^6+4a^2x^4\\
\qquad +2ax^6-4a^2x^4+8a^3x^2\\
\qquad\qquad\qquad +4a^2x^4-8a^3x^2+16a^4\\
\hline
x^8 \qquad\qquad +4a^2x^4 \qquad\qquad +16a^4
\end{array}$$

24.
$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc\\
a\ +b\ +c\\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc\\
\qquad +a^2b+b^3\ +bc^2\ -ab^2-abc-b^2c\\
\qquad\qquad\qquad +a^2c+b^2c+c^3-abc-ac^2-bc^2\\
\hline
a^3+b^3\ +c^3\ -3abc
\end{array}$$

25.
$$\begin{array}{rrrrrr}
x^2+4xy & +5y^2 & & & & \\
x^3-3x^2y & -2xy^2 & +3y^3 & & & \\
\hline
x^5+4x^4y & +5x^3y^2 & & & & \\
-3x^4y & -12x^3y^2 & -15x^2y^3 & & & \\
 & -2x^3y^2 & -8x^2y^3 & -10xy^4 & & \\
 & & +3x^2y^3 & +12xy^4 & +15y^5 & \\
\hline
x^5+x^4y & -9x^3y^2 & -20x^2y^3 & +2xy^4 & +15y^5 &
\end{array}$$

26.
$$\begin{array}{llll}
ab+cd+ac+bd & & & \\
ab+cd-ac-bd & & & \\
\hline
a^2b^2+abcd+a^2bc+ab^2d & & & \\
\quad +abcd+c^2d^2+ac^2d+bcd^2 & & & \\
\qquad\qquad -a^2bc-ac^2d-a^2c^2-abcd & & & \\
\qquad\qquad\qquad -ab^2d-bcd^2-abcd-b^2d^2 & & & \\
\hline
a^2b^2 & +c^2d^2 & -a^2c^2 & -b^2d^2
\end{array}$$

27. $(x-a)(x+a)(x^2+a^2)(x^4+a^4)=(x^2-a^2)(x^2+a^2)(x^4+a^4)$
$=(x^4-a^4)(x^4+a^4)=x^8-a^8.$

28. $(x-a)(x+b)(x-c)=(x^2-ax+bx-ab)(x-c)$
$=x^3-ax^2+bx^2-cx^2-abx+acx-bcx+abc.$

29. $(1-x)(1+x)(1+x^2)(1+x^4)=(1-x^2)(1+x^2)(1+x^4)$
$=(1-x^4)(1+x^4)=1-x^8.$

30. First multiply x^2+xy+y^2 by $x-y$; the result is x^3-y^3.
Then multiply x^2-xy+y^2 by $x+y$; the result is x^3+y^3.
Then multiply x^3-y^3 by x^3+y^3; the result is x^6-y^6.

31. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^2-x^2)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^4-x^4)(a^4+x^4)(a^8+x^8)$
$=(a^8-x^8)(a^8+x^8)=a^{16}-x^{16}.$

32. $(x-5)(x-6)(x+7)=(x^2-11x+30)(x+7)=x^3-4x^2-47x+210$; therefore the coefficient of x is -47.

33. $(x+8)(x+3)(x-2)=(x^2+11x+24)(x-2)=x^3+9x^2+2x-48$; therefore the coefficient of x is 2.

34. $(x-2)(x-3)(x+4)=(x^2-5x+6)(x+4)=x^3-x^2-14x+24$; therefore the coefficient of x is -14.

35. $(x-a)(x-b)(x-c)=(x^2-ax-bx+ab)(x-c)$
$=x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc$;
therefore the coefficient of x is $ab+ac+bc$.

36. $(x^2+3x-2)(x^2-3x+2)(x^4-5)=(x^4-9x^2+12x-4)(x^4-5)$
$=x^8-9x^6+12x^5-9x^4+45x^2-60x+20$;
therefore the coefficient of x is -60.

37. $(x^2-x+1)(x^2+x-1)(x^4-x^2+1)=(x^4-x^2+2x-1)(x^4-x^2+1)$
$=x^8-2x^6+2x^5+x^4-2x^3+2x-1$;
therefore the coefficient of x is 2.

38. $(x^2-mx+1)(x^2-mx-1)(x^4-m^2x-1)$
$=(x^4-2mx^3+m^2x^2-1)(x^4-m^2x-1)=x^8-2mx^7+m^2x^6-m^2x^5$
$+2m^3x^4-2x^4-m^4x^3+2mx^3-m^2x^2+m^2x+1$;
therefore the coefficient of x is m^2.

IX.

8.
$$\begin{array}{l}
-a^3-a^2-a \\
-a-1 \\
\hline
a^4+a^3+a^2 \\
\quad +a^3+a^2+a \\
\hline
a^4+2a^3+2a^2+a
\end{array}$$

9.
$$\begin{array}{l}
3x^2y-5xy^2+4y^3 \\
-2x-3y \\
\hline
-6x^3y+10x^2y^2-8xy^3 \\
\qquad\quad -9x^2y^2+15xy^3-12y^4 \\
\hline
-6x^3y+x^2y^2+7xy^3-12y^4
\end{array}$$

10.
$$\begin{array}{l}
-5m^2-6mn+7n^2 \\
-m+n \\
\hline
5m^3+6m^2n-7mn^2 \\
\qquad -5m^2n-6mn^2+7n^3 \\
\hline
5m^3+m^2n-13mn^2+7n^3
\end{array}$$

11.
$$\begin{array}{l}
13r^2-17r-45 \\
-r-3 \\
\hline
-13r^3+17r^2+45r \\
\qquad\quad -39r^2+51r+135 \\
\hline
-13r^3-22r^2+96r+135
\end{array}$$

12.
$$\begin{array}{r}
7x^3-8x^2z-9z^2 \quad\quad\quad\quad\quad\quad \\
-x-z \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \\
\hline
-7x^4+8x^3z+9xz^2 \quad\quad\quad\quad\quad\quad \\
-7x^3z+8x^2z^2+9z^3 \\
\hline
-7x^4+x^3z+8x^2z^2+9xz^2+9z^3
\end{array}$$

13.
$$\begin{array}{r}
-x^5+x^4y-x^3y^2 \quad\quad\quad \\
-x-y \quad\quad\quad\quad\quad\quad\quad\quad\quad \\
\hline
x^6-x^5y+x^4y^2 \quad\quad\quad\quad \\
+x^5y-x^4y^2+x^3y^3 \\
\hline
x^6 \quad\quad\quad\quad\quad\quad +x^3y^3
\end{array}$$

14.
$$\begin{array}{r}
-x^3-x^2y-xy^2-y^3 \quad\quad\quad \\
-x-y \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \\
\hline
x^4+x^3y+x^2y^2+xy^3 \quad\quad\quad \\
+x^3y+x^2y^2+xy^3+y^4 \\
\hline
x^4+2x^3y+2x^2y^2+2xy^3+y^4
\end{array}$$

X.

12. $(x^2+2x-3)^2=x^4+4x^2+9+4x^3-6x^2-12x=x^4+4x^3-2x^2-12x+9.$

13. $(x^2-6x+7)^2=x^4+36x^2+49-12x^3+14x^2-84x$
$=x^4-12x^3+50x^2-84x+49.$

14. $(2x^2-7x+9)^2=4x^4+49x^2+81-28x^3+36x^2-126x$
$=4x^4-28x^3+85x^2-126x+81.$

16. $(x^4-4x^2y^2+y^4)^2=x^8+16x^4y^4+y^8-8x^6y^2+2x^4y^4-8x^2y^6$
$=x^8-8x^6y^2+18x^4y^4-8x^2y^6+y^8.$

27. $(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$
$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$

28. $(a-b-c)^3=(a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3$
$=a^3-3a^2b+3ab^2-b^3-3a^2c+6abc-3b^2c+3ac^2-3bc^2-c^3.$

29. $(m+n)(m-n)(m+n)(m-n)=(m^2-n^2)(m^2-n^2)$
$=m^4-2m^2n^2+n^4.$

30. $(m^2+2mn+n^2)(m^2-n^2)=m^4+2m^3n+m^2n^2-m^2n^2-2mn^3-n^4$
$=m^4+2m^3n-2mn^3-n^4.$

XIV.

1.
$$\begin{array}{l} x+10)x^2+15x+50(x+5 \\ \quad \underline{x^2+10x} \\ \qquad\quad 5x+50 \\ \qquad\quad \underline{5x+50} \end{array}$$

2.
$$\begin{array}{l} x-7)x^2-17x+70(x-10 \\ \quad \underline{x^2-7x} \\ \qquad -10x+70 \\ \qquad \underline{-10x+70} \end{array}$$

3.
$$\begin{array}{l} x-3)x^2+x-12(x+4 \\ \quad \underline{x^2-3x} \\ \qquad\quad 4x-12 \\ \qquad\quad \underline{4x-12} \end{array}$$

4.
$$\begin{array}{l} x+1)x^2+13x+12(x+12 \\ \quad \underline{x^2+x} \\ \qquad\quad 12x+12 \\ \qquad\quad \underline{12x+12} \end{array}$$

5.
$$\begin{array}{l} x+6)x^3+13x^2+54x+72(x^2+7x+12 \\ \quad \underline{x^3+6x^2} \\ \qquad 7x^2+54x \\ \qquad \underline{7x^2+42x} \\ \qquad\qquad 12x+72 \\ \qquad\qquad \underline{12x+72} \end{array}$$

6.
$$\begin{array}{l} x+1)x^3+x^2-x-1(x^2-1 \\ \quad \underline{x^3+x^2} \\ \qquad\qquad -x-1 \\ \qquad\qquad \underline{-x-1} \end{array}$$

7.
$$\begin{array}{l} x+1)x^3+2x^2+2x+1(x^2+x+1 \\ \quad \underline{x^3+x^2} \\ \qquad x^2+2x \\ \qquad \underline{x^2+x} \\ \qquad\quad x+1 \\ \qquad\quad \underline{x+1} \end{array}$$

8.
$$\begin{array}{l} x^2+3x+1)x^5-5x^3+7x^2+6x+1(x^3-3x^2+3x+1 \\ \quad \underline{x^5+3x^4+x^3} \\ \qquad -3x^4-6x^3+7x^2 \\ \qquad \underline{-3x^4-9x^3-3x^2} \\ \qquad\qquad 3x^3+10x^2+6x \\ \qquad\qquad \underline{3x^3+9x^2+3x} \\ \qquad\qquad\qquad x^2+3x+1 \\ \qquad\qquad\qquad \underline{x^2+3x+1} \end{array}$$

9. $x^2 - 2x - 1)x^4 - 4x^3 + 2x^2 + 4x + 1(x^2 - 2x - 1$

$$\begin{array}{r}
\underline{x^4 - 2x^3 - x^2} \\
-2x^3 + 3x^2 + 4x \\
\underline{-2x^3 + 4x^2 + 2x} \\
-x^2 + 2x + 1 \\
\underline{-x^2 + 2x + 1}
\end{array}$$

10. $x^2 - 2x + 1)x^4 - 4x^3 + 6x^2 - 4x + 1(x^2 - 2x + 1$

$$\begin{array}{r}
\underline{x^4 - 2x^3 + x^2} \\
-2x^3 + 5x^2 - 4x \\
\underline{-2x^3 + 4x^2 - 2x} \\
x^2 - 2x - 1 \\
\underline{x^2 - 2x + 1}
\end{array}$$

11. $x^2 + x - 1)x^4 - x^2 + 2x - 1(x^2 - x + 1$

$$\begin{array}{r}
\underline{x^4 + x^3 - x^2} \\
-x^3 + 2x - 1 \\
\underline{-x^3 - x^2 + x} \\
x^2 + x - 1 \\
\underline{x^2 + x - 1}
\end{array}$$

12. $x + 2)x^4 - 4x^2 + 8x + 16(x^3 - 2x^2 + 8$

$$\begin{array}{r}
\underline{x^4 + 2x^3} \\
-2x^3 - 4x^2 \\
\underline{-2x^3 - 4x^2} \\
8x + 16 \\
\underline{8x + 16}
\end{array}$$

13. $$\begin{array}{l} x+4y)x^3+4x^2y+3xy^2+12y^3(x^2+3y^2 \\ \underline{x^3+4x^2y} \\ \qquad 3xy^2+12y^3 \\ \qquad \underline{3xy^2+12y^3} \end{array}$$

14. $$\begin{array}{l} a+b)a^4+4a^3b+6a^2b^2+4ab^3+b^4(a^3+3a^2b+3ab^2+b^3 \\ \underline{a^4+a^3b} \\ \quad 3a^3b+6a^2b^2 \\ \quad \underline{3a^3b+3a^2b^2} \\ \qquad 3a^2b^2+4ab^3 \\ \qquad \underline{3a^2b^2+3ab^3} \\ \qquad\quad ab^3+b^4 \\ \qquad\quad \underline{ab^3+b^4} \end{array}$$

15. $$\begin{array}{l} a-b)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5(a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\ \underline{a^5-a^4b} \\ \quad -4a^4b+10a^3b^2 \\ \quad \underline{-4a^4b+4a^3b^2} \\ \qquad 6a^3b^2-10a^2b^3 \\ \qquad \underline{6a^3b^2-6a^2b^3} \\ \qquad\quad -4a^2b^3+5ab^4 \\ \qquad\quad \underline{-4a^2b^3+4ab^4} \\ \qquad\qquad ab^4-b^5 \\ \qquad\qquad \underline{ab^4-b^5} \end{array}$$

16. $$\begin{array}{l} x^2-6x+9)x^4-12x^3+50x^2-84x+45(x^2-6x+5 \\ \underline{x^4-6x^3+9x^2} \\ \quad -6x^3+41x^2-84x \\ \quad \underline{-6x^3+36x^2-54x} \\ \qquad 5x^2-30x+45 \\ \qquad \underline{5x^2-30x+45} \end{array}$$

17. $$\begin{array}{l} a^2-2ab-3b^2)\,a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5\,(a^3-2a^2b+3ab^2+4b^3 \\ \underline{a^5-2a^4b-3a^3b^2} \\ -2a^4b+7a^3b^2+4a^2b^3 \\ \underline{-2a^4b+4a^3b^2+6a^2b^3} \\ 3a^3b^2-2a^2b^3-17ab^4 \\ \underline{3a^3b^2-6a^2b^3-9ab^4} \\ 4a^2b^3-8ab^4-12b^5 \\ \underline{4a^2b^3-8ab^4-12b^5} \end{array}$$

18. $$\begin{array}{l} 2ax^2-3a^2x+a^3)\,4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6\,(2ax^2-3a^2x+a^3 \\ \underline{4a^2x^4-6a^3x^3+2a^4x^2} \\ -6a^3x^3+11a^4x^2-6a^5x \\ \underline{-6a^3x^3+9a^4x^2-3a^5x} \\ 2a^4x^2-3a^5x+a^6 \\ \underline{2a^4x^2-3a^5x+a^6} \end{array}$$

19. $$\begin{array}{l} x^2+x-1)\,x^4-x^2+2x-1\,(x^2-x+1 \\ \underline{x^4+x^3-x^2} \\ -x^3+2x-1 \\ \underline{-x^3-x^2+x} \\ x^2+x-1 \\ \underline{x^2+x-1} \end{array}$$

20. $$\begin{array}{l} x^2+2a^2)\,x^4+a^2x^2-2a^4\,(x^2-a^2 \\ \underline{x^4+2a^2x^2} \\ -a^2x^2-2a^4 \\ \underline{-a^2x^2-2a^4} \end{array}$$

21. $$\begin{array}{l} x-15y)\,x^2-13xy-30y^2\,(x+2y \\ \underline{x^2-15xy} \\ 2xy-30y^2 \\ \underline{2xy-30y^2} \end{array}$$

22. $$\begin{array}{l} x+y)\,x^5+y^5\,(x^4-x^3y+x^2y^2-xy^3+y^4 \\ \underline{x^5+x^4y} \\ -x^4y+y^5 \\ \underline{-x^4y-x^3y^2} \\ x^3y^2+y^5 \\ \underline{x^3y^2+x^2y^3} \\ -x^2y^3+y^5 \\ \underline{-x^2y^3-xy^4} \\ xy^4+y^5 \\ \underline{xy^4+y^5} \end{array}$$

23. $$\begin{array}{l} x-y)\,x^6-y^6\,(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5 \\ \underline{x^6-x^5y} \\ x^5y-y^6 \\ \underline{x^5y-x^4y^2} \\ x^4y^2-y^6 \\ \underline{x^4y^2-x^3y^3} \\ x^3y^3-y^6 \\ \underline{x^3y^3-x^2y^4} \\ x^2y^4-y^6 \\ \underline{x^2y^4-xy^5} \\ xy^5-y^6 \\ \underline{xy^5-y^6} \end{array}$$

24. $$\begin{array}{l} a-b+c)\,a^2-b^2+2bc-c^2\,(a+b-c \\ \underline{a^2-ab+ac} \\ ab-b^2-ac+2bc-c^2 \\ \underline{ab-b^2+bc} \\ -ac+bc-c^2 \\ \underline{-ac+bc-c^2} \end{array}$$

25. $$\begin{array}{l} -1+b)\,b-3b^2+3b^3-b^4\,(-b+2b^2-b^3 \\ \underline{b-b^2} \\ -2b^2+3b^3 \\ \underline{-2b^2+2b^3} \\ b^3-b^4 \\ \underline{b^3-b^4} \end{array}$$

26. $$\begin{array}{l} a+b-c-d)\,a^2-2ad+2bc-b^2-c^2+d^2\,(a-b+c-d \\ \underline{a^2+ab-ac-ad} \\ -ab-b^2+ac-ad+2bc-c^2+d^2 \\ \underline{-ab-b^2+bc+bd} \\ ac+bc-bd-ad-c^2+d^2 \\ \underline{ac+bc-c^2-cd} \\ -ad-bd+cd+d^2 \\ \underline{-ad-bd+cd+d^2} \end{array}$$

27. $$x+y+z)\,x^3+y^3+z^3-3xyz\,(x^2-xy-xz+y^2-yz+z^2$$
$$\underline{x^3+x^2y+x^2z}$$
$$-x^2y-x^2z-3xyz+y^3+z^3$$
$$\underline{-x^2y-xy^2-xyz}$$
$$-x^2z+xy^2-2xyz+y^3+z^3$$
$$\underline{-x^2z-xyz-xz^2}$$
$$xy^2-xyz+xz^2+y^3+z^3$$
$$\underline{xy^2+y^3+y^2z}$$
$$-xyz-y^2z+xz^2+z^3$$
$$\underline{-xyz-y^2z-yz^2}$$
$$xz^2+yz^2+z^3$$
$$\underline{xz^2+yz^2+z^3}$$

28. $$x^3+y^2)\,x^{15}+y^{10}\,(x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8$$
$$\underline{x^{15}+x^{12}y^2}$$
$$-x^{12}y^2+y^{10}$$
$$\underline{-x^{12}y^2-x^9y^4}$$
$$x^9y^4+y^{10}$$
$$\underline{x^9y^4+x^6y^6}$$
$$-x^6y^6+y^{10}$$
$$\underline{-x^6y^6-x^3y^8}$$
$$x^3y^8+y^{10}$$
$$\underline{x^3y^8+y^{10}}$$

29. $$p-q+3r)\,p^2+pq+2pr-2q^2+7qr-3r^2\,(p+2q-r$$
$$\underline{p^2-pq+3pr}$$
$$2pq-pr-2q^2+7qr$$
$$\underline{2pq-2q^2+6qr}$$
$$-pr+qr-3r^2$$
$$\underline{-pr+qr-3r^2}$$

30. $$\begin{array}{l} a^4 + a^3b + a^2b^2 + ab^3 + b^4)\, a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8\,(a^4 - a^3b + a^2b^2 - ab^3 + b^4 \\ \underline{a^8 + a^7b + a^6b^2 + a^5b^3 + a^4b^4} \\ -a^7b - a^5b^3 + a^2b^6 + b^8 \\ \underline{-a^7b - a^6b^2 - a^5b^3 - a^4b^4 - a^3b^5} \\ a^6b^2 + a^4b^4 + a^3b^5 + a^2b^6 + b^8 \\ \underline{a^6b^2 + a^5b^3 + a^4b^4 + a^3b^5 + a^2b^6} \\ -a^5b^3 + b^8 \\ \underline{-a^5b^3 - a^4b^4 - a^3b^5 - a^2b^6 - ab^7} \\ a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8 \\ \underline{a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8} \end{array}$$

31. $$\begin{array}{l} x^4 - x^3y + x^2y^2 - xy^3 + y^4)\, x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8\,(x^4 + x^3y + x^2y^2 + xy^3 + y \\ \underline{x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4} \\ x^7y + x^5y^3 + x^2y^6 + y^8 \\ \underline{x^7y - x^6y^2 + x^5y^3 - x^4y^4 + x^3y^5} \\ x^6y^2 + x^4y^4 - x^3y^5 + x^2y^6 + y^8 \\ \underline{x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6} \\ x^5y^3 + y^8 \\ \underline{x^5y^3 - x^4y^4 + x^3y^5 - x^2y^6 + xy^7} \\ x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8 \\ \underline{x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8} \end{array}$$

32. $$\begin{array}{l} 2x^2 + 3x + 2)\, 4x^5 - x^3 + 4x\,(2x^3 - 3x^2 + 2x \\ \underline{4x^5 + 6x^4 + 4x^3} \\ -6x^4 - 5x^3 + 4x \\ \underline{-6x^4 - 9x^3 - 6x^2} \\ 4x^3 + 6x^2 + 4x \\ \underline{4x^3 + 6x^2 + 4x} \end{array}$$

$$\begin{array}{l}
a-3)\,a^5-243\,(a^4+3a^3+9a^2+27a+81\\
\qquad \underline{a^5-3a^4}\\
\qquad\qquad 3a^4-243\\
\qquad\qquad \underline{3a^4-9a^3}\\
\qquad\qquad\qquad 9a^3-243\\
\qquad\qquad\qquad \underline{9a^3-27a^2}\\
\qquad\qquad\qquad\qquad 27a^2-243\\
\qquad\qquad\qquad\qquad \underline{27a^2-81a}\\
\qquad\qquad\qquad\qquad\qquad 81a-243\\
\qquad\qquad\qquad\qquad\qquad \underline{81a-243}
\end{array}$$

34. $$\begin{array}{l}
k^3-1)\,k^{10}-k\,(k^7+k^4+k\\
\qquad \underline{k^{10}-k^7}\\
\qquad\qquad k^7-k\\
\qquad\qquad \underline{k^7-k^4}\\
\qquad\qquad\qquad k^4-k\\
\qquad\qquad\qquad \underline{k^4-k}
\end{array}$$

35. $$\begin{array}{l}
x+4)\,x^3-5x^2-46x-40\,(x^2-9x-10\\
\qquad \underline{x^3+4x^2}\\
\qquad\qquad -9x^2-46x\\
\qquad\qquad \underline{-9x^2-36x}\\
\qquad\qquad\qquad -10x-40\\
\qquad\qquad\qquad \underline{-10x-40}
\end{array}$$

36. $$\begin{array}{l}
2x-3a)\,48x^3-76ax^2-64a^2x+105a^3\,(24x^2-2ax-35a^2\\
\qquad \underline{48x^3-72ax^2}\\
\qquad\qquad -4ax^2-64a^2x\\
\qquad\qquad \underline{-4ax^2+6a^2x}\\
\qquad\qquad\qquad -70a^2x+105a^3\\
\qquad\qquad\qquad \underline{-70a^2x+105a^3}
\end{array}$$

37. $$\begin{array}{l}
3x^2-4x+5)\,18x^4-45x^3+82x^2-67x+40\,(6x^2-7x+8\\
\qquad \underline{18x^4-24x^3+30x^2}\\
\qquad\qquad -21x^3+52x^2-67x\\
\qquad\qquad \underline{-21x^3+28x^2-35x}\\
\qquad\qquad\qquad 24x^2-32x+40\\
\qquad\qquad\qquad \underline{24x^2-32x+40}
\end{array}$$

38. $$\begin{array}{l}2x-3a)16x^4-72a^2x^2+81a^4(8x^3+12ax^2-18a^2x-27a^3\\ \underline{16x^4-24ax^3}\\ 24ax^3-72a^2x^2\\ \underline{24ax^3-36a^2x^2}\\ -36a^2x^2+81a^4\\ \underline{-36a^2x^2+54a^3x}\\ -54a^3x+81a^4\\ \underline{-54a^3x+81a^4}\end{array}$$

39. $$\begin{array}{l}3x+4a)81x^4-256a^4(27x^3-36ax^2+48a^2x-64a^3\\ \underline{81x^4+108ax^3}\\ -108ax^3-256a^4\\ \underline{-108ax^3-144a^2x^2}\\ 144a^2x^2-256a^4\\ \underline{144a^2x^2+192a^3x}\\ -192a^3x-256a^4\\ \underline{-192a^3x-256a^4}\end{array}$$

40. $$\begin{array}{l}a^2-b^2)2a^3+3a^2b-2ab^2-3b^3(2a+3b\\ \underline{2a^3-2ab^2}\\ 3a^2b-3b^3\\ \underline{3a^2b-3b^3}\end{array}$$

41. $$\begin{array}{l}x^2-a^2)x^3+2ax^2-a^2x-2a^3(x+2a\\ \underline{x^3-a^2x}\\ 2ax^2-2a^3\\ \underline{2ax^2-2a^3}\end{array}$$

42. $$\begin{array}{l}a^2+3b^2)a^4-a^2b^2-12b^4(a^2-4b^2\\ \underline{a^4+3a^2b^2}\\ -4a^2b^2-12b^4\\ \underline{-4a^2b^2-12b^4}\end{array}$$

43. $x^2+3x+y)x^4-9x^2-6xy-y^2(x^2-3x-y$

$$\underline{x^4+3x^3+x^2y}$$

$$-3x^3-x^2y-9x^2-6xy-y^2$$

$$\underline{-3x^3-9x^2-3xy}$$

$$-x^2y-3xy-y^2$$

$$\underline{-x^2y-3xy-y^2}$$

44. $x^2-3xy+2y^2)x^4-6x^3y+9x^2y^2-4y^4(x^2-3xy-2y^2$

$$\underline{x^4-3x^3y+2x^2y^2}$$

$$-3x^3y+7x^2y^2-4y^4$$

$$\underline{-3x^3y+9x^2y^2-6xy^3}$$

$$-2x^2y^2+6xy^3-4y^4$$

$$\underline{-2x^2y^2+6xy^3-4y^4}$$

45. $x-3y)x^4-81y^4(x^3+3x^2y+9xy^2+27y^3$

$$\underline{x^4-3x^3y}$$

$$3x^3y-81y^4$$

$$\underline{3x^3y-9x^2y^2}$$

$$9x^2y^2-81y^4$$

$$\underline{9x^2y^2-27xy^3}$$

$$27xy^3-81y^4$$

$$\underline{27xy^3-81y^4}$$

46. $a-2b)a^4-16b^4(a^3+2a^2b+4ab^2+8b^3$

$$\underline{a^4-2a^3b}$$

$$2a^3b-16b^4$$

$$\underline{2a^3b-4a^2b^2}$$

$$4a^2b^2-16b^4$$

$$\underline{4a^2b^2-8ab^3}$$

$$8ab^3-16b^4$$

$$\underline{8ab^3-16b^4}$$

47. $$\begin{array}{l}
3a + 2b)81a^4 - 16b^4(27a^3 - 18a^2b + 12ab^2 - 8b^3 \\
\qquad\qquad \underline{81a^4 + 54a^3b} \\
\qquad\qquad\qquad -54a^3b - 16b^4 \\
\qquad\qquad\qquad \underline{-54a^3b - 36a^2b^2} \\
\qquad\qquad\qquad\qquad\qquad 36a^2b^2 - 16b^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{36a^2b^2 + 24ab^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad -24ab^3 - 16b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-24ab^3 - 16b^4}
\end{array}$$

48. $$\begin{array}{l}
2x + 3y)16x^4 - 81y^4(8x^3 - 12x^2y + 18xy^2 - 27y^3 \\
\qquad\qquad \underline{16x^4 + 24x^3y} \\
\qquad\qquad\qquad -24x^3y - 81y^4 \\
\qquad\qquad\qquad \underline{-24x^3y - 36x^2y^2} \\
\qquad\qquad\qquad\qquad\qquad 36x^2y^2 - 81y^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{36x^2y^2 + 54xy^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad -54xy^3 - 81y^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-54xy^3 - 81y^4}
\end{array}$$

49. $$\begin{array}{l}
a + 2b + 3c)3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2(3a + 2b + c \\
\qquad\qquad\qquad \underline{3a^2 + 6ab + 9ac} \\
\qquad\qquad\qquad\qquad 2ab + 4b^2 + ac + 8bc \\
\qquad\qquad\qquad\qquad \underline{2ab + 4b^2 + 6bc} \\
\qquad\qquad\qquad\qquad\qquad\qquad ac + 2bc + 3c^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac + 2bc + 3c^2}
\end{array}$$

50. $$\begin{array}{l}
a^2 + 2ax + 4x^2)a^4 + 4a^2x^2 + 16x^4(a^2 - 2ax + 4x^2 \\
\qquad\qquad\qquad \underline{a^4 + 2a^3x + 4a^2x^2} \\
\qquad\qquad\qquad\quad -2a^3x + 16x^4 \\
\qquad\qquad\qquad\quad \underline{-2a^3x - 4a^2x^2 - 8ax^3} \\
\qquad\qquad\qquad\qquad\qquad 4a^2x^2 + 8ax^3 + 16x^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{4a^2x^2 + 8ax^3 + 16x^4}
\end{array}$$

51. $x^2 - xy + y^2)\, x^4 + x^2y^2 + y^4\, (x^2 + xy + y^2$
$\underline{x^4 - x^3y + x^2y^2}$
$x^3y + y^4$
$\underline{x^3y - x^2y^2 + xy^3}$
$x^2y^2 - xy^3 + y^4$
$\underline{x^2y^2 - xy^3 + y^4}$

52. $16x^2 + 4xy + y^2)\, 256x^4 + 16x^2y^2 + y^4\, (16x^2 - 4xy + y^2$
$\underline{256x^4 + 64x^3y + 16x^2y^2}$
$-64x^3y + y^4$
$\underline{-64x^3y - 16x^2y^2 - 4xy^3}$
$16x^2y^2 + 4xy^3 + y^4$
$\underline{16x^2y^2 + 4xy^3 + y^4}$

53. $x^3 + x - y)\, x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3\, (x^2 + xy - y^2$
$\underline{x^5 + x^3 - x^2y}$
$x^4y - x^3y^2 + x^2y - 2xy^2 + y^3$
$\underline{x^4y + x^2y - xy^2}$
$-x^3y^2 - xy^2 + y^3$
$\underline{-x^3y^2 - xy^2 + y^3}$

54. $x - a)\, ax^3 + 3a^2x^2 - 2a^3x - 2a^4\, (ax^2 + 4a^2x + 2a^3$
$\underline{ax^3 - a^2x^2}$
$4a^2x^2 - 2a^3x$
$\underline{4a^2x^2 - 4a^3x}$
$2a^3x - 2a^4$
$\underline{2a^3x - 2a^4}$

55. $a + x)\, a^2 - x^2\, (a - x$
$\underline{a^2 + ax}$
$-ax - x^2$
$\underline{-ax - x^2}$

56. $2x + 3y + z)\, 2x^2 + xy - 3y^2 - 4yz - xz - z^2\, (x - y - z$
$\underline{2x^2 + 3xy + xz}$
$-2xy - 3y^2 - 4yz - 2xz - z^2$
$\underline{-2xy - 3y^2 - yz}$
$-2xz - 3yz - z^2$
$\underline{-2xz - 3yz - z^2}$

57. $$\begin{array}{l} 1+5x+x^2)\,2+9x+14x^3+3x^4\,(2-x+3x^2 \\ \quad\underline{2+10x+2x^2} \\ \quad -x-2x^2+14x^3 \\ \quad \underline{-x-5x^2-x^3} \\ \qquad 3x^2+15x^3+3x^4 \\ \qquad \underline{3x^2+15x^3+3x^4} \end{array}$$

58. $$\begin{array}{l} 3-5x+7x^2)\,12-38x+82x^2-112x^3+106x^4-70x^5\,(4-6x+8x^2-10x^3 \\ \quad\underline{12-20x+28x^2} \\ \quad -18x+54x^2-112x^3 \\ \quad \underline{-18x+30x^2-42x^3} \\ \qquad 24x^2-70x^3+106x^4 \\ \qquad \underline{24x^2-40x^3+56x^4} \\ \qquad\quad -30x^3+50x^4-70x^5 \\ \qquad\quad \underline{-30x^3+50x^4-70x^5} \end{array}$$

59. $$\begin{array}{l} x^4-x^3y+x^2y^2-xy^3+y^4)\,x^5+y^5\,(x+y \\ \quad\underline{x^5-x^4y+x^3y^2-x^2y^3+xy^4} \\ \qquad x^4y-x^3y^2+x^2y^3-xy^4+y^5 \\ \qquad \underline{x^4y-x^3y^2+x^2y^3-xy^4+y^5} \end{array}$$

60. $$\begin{array}{l} ax+xy+ab+by)\,a^2x^2-a^2b^2-x^2y^2+b^2y^2\,(ax-xy-ab+by \\ \quad\underline{a^2x^2+ax^2y+a^2bx+abxy} \\ \quad -ax^2y-a^2bx-abxy-a^2b^2-x^2y^2+b^2y^2 \\ \quad \underline{-ax^2y-x^2y^2-abxy-bxy^2} \\ \qquad -a^2bx+bxy^2-a^2b^2+b^2y^2 \\ \qquad \underline{-a^2bx-abxy-a^2b^2-ab^2y} \\ \qquad\quad abxy+bxy^2+ab^2y+b^2y^2 \\ \qquad\quad \underline{abxy+bxy^2+ab^2y+b^2y^2} \end{array}$$

61.
$$\begin{array}{l}
ax+by)\,abx^2+a^2xy+b^2xy+aby^2\,(bx+ay\\
\quad\underline{abx^2+b^2xy}\\
\qquad\qquad a^2xy+aby^2\\
\qquad\qquad \underline{a^2xy+aby^2}
\end{array}$$

62.
$$\begin{array}{l}
x^2+ax+b^2)\,x^4+2b^2x^2-a^2x^2+b^4\,(x^2-ax+b^2\\
\qquad\underline{x^4+ax^3+b^2x^2}\\
\qquad\quad -ax^3+b^2x^2-a^2x^2\\
\qquad\quad \underline{-ax^3-a^2x^2-ab^2x}\\
\qquad\qquad\qquad b^2x^2+ab^2x+b^4\\
\qquad\qquad\qquad \underline{b^2x^2+ab^2x+b^4}
\end{array}$$

XV.

1.
$$\begin{array}{l}
x^2-ax+c)\,x^4-(a^2-b-c)x^2-(b-c)ax+bc\,(x^2+ax+b\\
\qquad\underline{x^4-ax^3+cx^2}\\
\qquad\quad \underline{ax^3-(a^2-b)x^2-(ab-ac)x+bc}\\
\qquad\quad \underline{ax^3-a^2x^2+acx}\\
\qquad\qquad bx^2-abx+bc\\
\qquad\qquad \underline{bx^2-abx+bc}
\end{array}$$

2.
$$\begin{array}{l}
y-n)\,y^3-(l+m+n)y^2+(lm+ln+mn)y-lmn\,(y^2-(l+m)y+lm\\
\quad\underline{y^3-ny^2}\\
\qquad -(l+m)y^2+(lm+ln+mn)y-lmn\\
\qquad \underline{-(l+m)y^2+(ln+mn)y}\\
\qquad\qquad\qquad lmy-lmn\\
\qquad\qquad\qquad \underline{lmy-lmn}
\end{array}$$

3. $$x^3-mx^2+nx+r)x^5-(m-c)x^4+(n-cm+d)x^3+(r+cn-dm)x^2+$$
$$(cr+dn)x+dr(x^2+cx+d$$
$$\underline{x^5-mx^4+nx^3+rx^2}$$
$$cx^4+(d-cm)x^3+(cn-dm)x^2$$
$$\underline{cx^4-cmx^3+cnx^2+crx}$$
$$dx^3-dmx^2+dnx+dr$$
$$\underline{dx^3-dmx^2+dnx+dr}$$

4. $$x^2+5x-4)x^4+(5+a)x^3-(4-5a+b)x^2-(4a+5b)x+4b(x^2+ax-b$$
$$\underline{x^4+5x^3-4x^2}$$
$$ax^3+(5a-b)x^2-(4a+5b)x$$
$$\underline{ax^3+5ax^2-4ax}$$
$$-bx^2-5bx+4b$$
$$\underline{-bx^2-5bx+4b}$$

5. $$x^2-(a+c)x+ac)x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$-(abc+abd+acd+bcd)x+abcd(x^2-(b+d)x+bd$$
$$\underline{x^4-(a+c)x^3+acx^2}$$
$$-(b+d)x^3+(ab+ad+bc+bd+cd)x^2$$
$$\underline{-(b+d)x^3+(ab+ad+bc+cd)x^2-(abc+acd)x}$$
$$bdx^2-(abd+bcd)x+abcd$$
$$\underline{bdx^2-(abd+bcd)x+abcd}$$

XVIII.

1. $(x^2-ax)-(bx-ab)=x(x-a)-b(x-a)=(x-b)\,(x-a)$.

2. $(ab+ax)-(bx+x^2)=a(b+x)-x(b+x)=(a-x)\,(b+x)$.

3. $(bc+by)-(cy+y^2)=b(c+y)-y(c+y)=(b-y)\,(c+y)$.

4. $(bm+mn)+(ab+an)=m(b+n)+a(b+n)=(a+m)\,(b+n)$.

5. $(abx^2 - axy) + (bxy - y^2) = ax(bx - y) + y(bx - y) = (ax + y)(bx - y)$.

6. $(abx - aby) + (cdx - cdy) = ab(x - y) + cd(x - y) = (ab + cd)(x - y)$.

7. $(cdx^2 + dmxy) - (cnxy + mny^2) = dx(cx + my) - ny(cx + my)$
$= (cx + my)(dx - ny)$.

8. $(abcx - b^2dx) - (acdy - bd^2y) = bx(ac - bd) - dy(ac - bd)$
$= (ac - bd)(bx - dy)$.

XXVI.

15. $(x^2 - 2xy + y^2) - z^2 = (x - y)^2 - z^2 = (x - y + z)(x - y - z)$.

17. $(a^2 - 2ac + c^2) - (b^2 + 2bd + d^2) = (a - c)^2 - (b + d)^2$, etc.

18. $a^2 - (b^2 - 2bc + c^2) = a^2 - (b - c)^2$, etc.

19. $(x^2 + 2xy + y^2) - z^2 = (x + y)^2 - z^2$, etc.

20. $(a^2 - 2ab + b^2) - (m^2 - 2mn + n^2) = (a - b)^2 - (m - n)^2$, etc.

23. $1 - (a^2 - 2ab + b^2) = 1 - (a - b)^2$, etc.

24. $1 - (x^2 - 2xy + y^2) = 1 - (x - y)^2$, etc.

25. $x^2 - (y^2 + 2yz + z^2) = x^2 - (y + z)^2$, etc.

26. $a^2 - (4b^2 - 12bc + 9c^2) = a^2 - (2b - 3c)^2$, etc.

29. $(a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2$, etc.

30. $(a^2 - 2ac + c^2) - (b^2 - 2bd + d^2) = (a - c)^2 - (b - d)^2$, etc.

31. $3ax(a^2x^2 - 9) = 3ax(ax + 3)(ax - 3)$.

XXVII.

11. $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

12. $x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$.

13. $a^6-64=a^6-2^6=(a^3+2^3)(a^3-2^3)$
$=(a+2)(a^2-2a+4)(a-2)(a^2+2a+4)$.

14. $729-y^6=3^6-y^6=(3^3+y^3)(3^3-y^3)$
$=(3+y)(9-3y+y^2)(3-y)(9+3y+y^2)$

XXX.

1. $6+4-5-3=2$.

2. $6+3-5-4=0$.

3. $18+12-5-8=17$.

4. $3\times 11-2\times 1=33-2=31$.

5. $10\times 2=20$.

6. $12+3\times 7=12+21=33$.

7. $(12+3)(4+3)=15\times 7=105$.

8. $12+12+3=27$.

9. $(25+3)\div(6-4)=28\div 2=14$.

10. $6\times 5\times 4=120$.

11. $6\times 5\times 7=210$.

12. $6\times 3\times 9\times 9=1458$.

13. $6\times 5\times 1\times 1=30$.

14. $\sqrt{5\times 5}=5$.

15. $\sqrt{y^2}=y=3$.

16. $(\sqrt{x})^2=(2)^2=4$.

17. $(\sqrt{x}+b)^2=(2+5)^2=49$.

18. $\sqrt{5\times 5\times 4}=\sqrt{100}=10$.

19. $\sqrt{2 \times 6 \times 4 \times 3} = \sqrt{144} = 12$.

20. $(36+25+3) \div (4+9+3) = 64 \div 16 = 4$.

21. $18 + (8-3)^2 = 18 + 25 = 43$.

22. $\{6-(5-3)\}\{6-(4-3)\} = 4 \times 5 = 20$.

23. $(6-5-3)^2 + (6-4+3)^2 = 4 + 25 = 29$.

24. $3 \times 8^3 + 4 \times 10^4 = 1536 + 40000$, etc.

25. $3 \times 1^2 + 7^2 = 3 + 49 = 52$.

XXXI.

1. $3 \times 3 \times 2 \times 1 - 3^3 + 2^3 + 1^3 = 18 - 27 + 8 + 1 = 0$.

2. $3^3 + 2^3 - 5^3 + 3 \times 3 \times 2 \times 5 = 27 + 8 - 125 + 90 = 0$.

3. $(a^2 + 2ac + c^2) - (a^2 + c^2) = 2ac$.

4. $x^2 + y^2 - (x^2 - 2xy + y^2) = 2xy$.

5. $(a^2 + 2ab + b^2)x - (a^2 + 2abx + b^2x^2) = (a^2 + b^2)x - a^2 - b^2x^2$.

6. Multiply $2x - m$ by $x + 2m$; the result is $2x^2 + 3mx - 2m^2$.
Multiply $2x + n$ by $x - 2n$; the result is $2x^2 - 3nx - 2n^2$.
Then multiply the two results together.

7.
$$
\begin{array}{l}
ar + b\big)\,acr^3 + (bc+ad)r^2 + (bd+ae)r + be\,\big(cr^2 + dr + e \\
\qquad\quad \underline{acr^3 + bcr^2} \\
\qquad\qquad\qquad adr^2 + (bd+ae)r \\
\qquad\qquad\qquad \underline{adr^2 + bdr} \\
\qquad\qquad\qquad\qquad\quad aer + be \\
\qquad\qquad\qquad\qquad\quad \underline{aer + be}
\end{array}
$$

The divisor $= 10 + 1$, or 11; the dividend $= 1000 + 200 + 20 + 1$, or 1221; the quotient $= 100 + 10 + 1$, or 111; and $1221 \div 11 = 111$.

8. Multiply $a+b+c$ by $a+b-c$; the result is $(a+b)^2-c^2$, or $a^2+2ab+b^2-c^2$.

Multiply $c+b-a$ by $c+a-b$; the result is $c^2-(a-b)^2$, or $c^2-a^2+2ab-b^2$.

Then $(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)=(2ab)^2-(a^2+b^2-c^2)^2$

$$=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2$$
$$=-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2.$$

9. $(a+d)(d+c)-(c+d)(d+a)-(a+c)(d-d)$

$$=ad+ac+d^2+cd-cd-ac-d^2-ad-0=0.$$

10. $0+(4-16)+\{16-(0+6)\}+\{12-(0+6)\}^2$

$$=-12+10+6^2=-2+36=34.$$

11. $\{4-3+2-1\}\{4+3-2-1\}=2\times4=8$;

and $16-9-4+1+2\times2=8$.

18. $(ac-ad-bc+bd)-(bc-bd-ac+ad)=2ac-2ad-2bc+2bd.$

19. $ab+ay+bx+xy+x-y+ax-by+ax+ay=$ etc.

20. $(x^2+3x+2)-(x^2-3x+2)=6x$, etc.

21. $ax-by+x-y+x^2-xy+ab-ay-bx+xy$, etc.

22. $(6x^2+20x+16)-(6x^2-28x+16)=48x$, etc.

23. $2mx-3ny+x+y+4mx+4nx-4my-4ny+mx+ny=$ etc.

24. $x^2+y^2+z^2+2xy+2xz+2yz+x^2+y^2+z^2$

$$=(x^2+2xy+y^2)+(y^2+2yz+z^2)+(x^2+2xz+z^2)$$
$$=(x+y)^2+(y+z)^2+(x+z)^2.$$

25. $4a^2+6ac-4ab+6ab+9bc-6b^2=4a^2+6ac+2ab+9bc-6b^2.$

26. $\dfrac{ab-cd}{cd+e}=\dfrac{63-15}{15+1}=\dfrac{48}{16}=3.$

$(bc-ad)(bd-ce)=(35-27)(21-5)=8\times16=128.$

$\dfrac{b^2-c^2}{c+d}=\dfrac{49-25}{5+3}=\dfrac{24}{8}=3.$

$d^c-c^d=3^5-5^3=243-125=118.$

27. $0-0+2^3+1^3=8+1=9.$

28. $48+4-8=44.$

29. $(1-2-3)^2+(2-1-3)^2+(3-1-2)^2=(-4)^2+(-2)^2+0=16+4$
$=20.$

30. $(1+2-4)^2+(1-2+4)^2+(2+4-1)^2=(-1)^2+(3)^2+(5)^2$
$=1+9+25=35.$

31. $(-1+2)^2+(2-3)^2+(-1-3)^2=(1)^2+(-1)^2+(-4)^2=1+1+16$
$=18.$

32. Let x and y be the numbers; then $(x^2-y^2)\div(x+y)=x-y.$

33. Let x and y be the numbers; then $(x+y)(x-y)=x^2-y^2.$

34. Let x and $x+1$ be the integers; then $(x+x+1)^2=4x^2+4x+1$
$=4x(x+1)+1.$

35. Let x and $x+2$ be the two even numbers; then $x+1$ is the odd number between them; and $(x+x+2)^2=4x^2+8x+4=4(x+1)^2.$

36. Let x and y be the parts; then $x+y=2$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=2(x-y).$

37. Let x and y be the parts; then $x+y=50$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=50(x-y).$

38. Let x and y be the parts; then $x+y=n$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=n(x-y).$

39. Let x and $x+1$ be the numbers; then $x(x+1)+x^2+(x+1)^2$
$=x^2+x+x^2+x^2+2x+1=3x^2+3x+1=(x+1)^3-x^3.$

40. Let $x-1, x, x+1$ be the three numbers; then $(x-1)^3+x^3+(x+1)^3$
$=x^3-3x^2+3x-1+x^3+x^3+3x^2+3x+1=3x^3+6x=3x(x^2+2).$

XXXII.

1. $7x-5x=11-5$; $2x=6$; $x=3$.
2. $12x-8x=15-7$; $4x=8$; $x=2$.
3. $236x-97x=564-425$; $139x=139$; $x=1$.
4. $5x-3x=7+7$; $2x=14$; $x=7$.
5. $12x-8x=9-1$; $4x=8$; $x=2$.
6. $124x-112x=43-19$; $12x=24$; $x=2$.
7. $5x-2x=27-18$; $3x=9$; $x=3$.
8. $12x-7x=145-125$; $5x=20$; $x=4$.
9. $14x-8x=80-26$; $6x=54$; $x=9$.
10. $-3x-x=-83-133$; $-4x=-216$; $4x=216$; $x=54$.
11. $-3x-5x=-3-13$; $-8x=-16$; $8x=16$; $x=2$.
12. $9x-12x=100-127$; $-3x=-27$; $3x=27$; $x=9$.
13. $-5x+4x=6-15$; $-x=-9$; $x=9$.
14. $3x-7x=6+22$; $-4x=28$; $4x=-28$; $x=-7$.
15. $4x-12x=-16-8$; $-8x=-24$; $8x=24$; $x=3$.
16. $5x-3x+7=4x-6x+35$; $4x=28$; $x=7$.
17. $6x-18+8x+15x-21=10x-4-16x+35$; $35x=70$; $x=2$.
18. $9x-15x+18+30=0$; $-6x=-48$; $x=8$.
19. $12x-45x-15+42-48x+783=0$; $-81x=-810$; $x=10$.
20. $x-28x+77=14x-70-152+19x-61$; $-60x=-360$; $x=6$.
21. $x^2+4x-21=x^2-20x+75$; $24x=96$; $x=4$.

22. $x^2+4x-96=x^2-5x-6$; $9x=90$; $x=10$.

23. $9x-x^2-14+x^2-2x-15-2x+2+12=0$; $5x=15$; $x=3$.

24. $2x^2+3x-35=36-17x+2x^2+229$; $20x=300$; $x=15$.

25. $21-32x+12x^2=12x^2-17x+6$; $-15x=-15$; $x=1$.

26. $14-x-5x^2+5x+30+20-29x+5x^2=45x-76$; $-70x=-140$; $x=2$.

27. $x^2+10x+25-16+8x-x^2=21x$; $-3x=-9$; $x=3$.

28. $5x^2-20x+20+7x^2-42x+63=12x^2-85x+133+42$; $23x=92$; $x=4$.

29. $9x^2-102x+289+16x^2-200x+625-25x^2+290x-841=1$; $-12x=-72$; $x=6$.

30. $x^2-4x-45+x^2+2x-80=2x^2-11x-21-113$; $9x=-9$; $x=-1$.

XXXIII.

1. Let x be the number; then $2x+14=154$; $x=70$.

2. Let x be the number; then $4x+16=188$; $x=43$.

3. Let x be the number; then $x+46=3x$; $x=23$.

4. Let x be the smaller number; then $3x$ is the greater number; and $16-x=30-3x$; $2x=14$; $x=7$; $3x=21$.

5. Let x be the first part; then $x-10$ is the second; $x-18$ the third; $x-24$ the fourth; and $x+x-10+x-18+x-24=92$; $x=36$, etc.

6. Let x be the greater number; then $20-x$ is the smaller number; and $3(20-x)+5x=84$; $x=12$, etc.

7. Let x be the father's age in years; then $80-x$ is the son's age; and $2(80-x)=x+10$; $x=50$, etc.

8. Let x be the age of the eldest; then $x-20$ is the age of the youngest; and $x=3(x-20)$; $x=30$, etc.

9. Let x be the sum in pounds; then $x+24-80=80-x$; $x=68$.

10. Let x be the price of a yard of cloth in shillings; then $2x$ is the price of a yard of silk; and $30x+80x=66\times 20$; $x=12$, etc.

11. Let x be the number; then $2x+24-80=100-x$; $x=52$.

12. Let x be A's share in pounds; then B's share is $280-x$; C's, $260-x$; D's, $220-x$; and $x+280-x+260-x+220-x=500$; $x=130$, etc.

13. Let x be the number of children; then there are $2x$ women and $4x$ men; and $x+2x+4x=266$; $x=38$, etc.

14. Let x be B's share in pounds; then A's share is $x-100$, and C's is $x+270$; and $x+x-100+x+270=1520$; $x=450$, etc.

15. Let x be the greater; then $x-8$ is the less; and $4(x-8)=2x+10$; $x=21$; $x-8=13$.

16. Let x be what each had at first in pounds; then $3(x+5)=11(x-5)$; $x=$ £8, 15s.

17. Let x be A's age; then $x-58$ is B's age; and $x-60=50-(x-58)$; $x=84$, etc.

18. Let x be A's age; then $x-34$ is B's age; and $x-50=40-(x-34)$; $x=62$, etc.

19. Let x be the share of a daughter; then $2x$ is the share of a son; and $3x+4x+500$ is the share of the wife; then $3x+4x+3x+4x+500=7500$; $x=500$, etc.

20. Let x be the number of gallons in the vessel at first; then $x+42=7x$; $x=7$; and $42+7=49$, the number of gallons that the vessel held.

21. Let x be the number of pounds A has; then B has $x+10$; C has $x+x+10$; and $x+x+10+x+x+10=76$; $x=14$, etc.

22. Let x be the greater; then $x-14$ is the less; and $x+x-14=48$; $x=31$, etc.

23. Let x be the number of pounds won by A; then $72+x=3(52-x)$; $x=21$.

24. Let x be one of the parts; then $84-x$ is the other; and $3x=4(84-x)$; $x=48$.

25. Let x be one of the parts; then $90-x$ is the other; and $4x=5(90-x)$; $x=50$.

26. Let x be the greater part; then $60-x$ is the less; and $x=60-x+24$; $x=42$.

27. Let x be the greater part; then $84-x$ is the less; and $x-36=84-x$; $x=60$.

28. Let x be one of the parts; then $20-x$ is the other; and $3x+5(20-x)=84$; $x=8$.

29. Let x be B's age; then $2x$ is A's age; and $2x-22=3(x-22)$; $x=44$; $2x=88$.

30. Let x be the number of years; then $30+x=2(6+x)$; $x=18$.

31. Let x be B's age; then $2x$ is A's age; and $2x-20=3(x-20)$; $x=40$.

32. Let x be B's age; then $3x$ is A's age; and $3x+19=2(x+19)$; $x=19$.

33. Let x be the number of years; then $50+x=42+3x$; $x=4$.

34. Let x be the number of guineas; then $x+48$ is the number of half-crowns; and, expressing all the quantities as sixpences, $42x+5(x+48)=100\times 40$; $47x=3760$; $x=80$.

35. Let x be the number of shillings; then $41 - x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x + 5(41 - x) = 74 \times 2$; $-3x = -57$; $x = 19$.

36. Let x be the number of shillings; then $300 - x$ is the number of fourpenny pieces; and, expressing all the quantities as fourpenny pieces, $3x + 300 - x = 700$; $x = 200$.

37. Let x be the number of moidores; then $x + 3$ is the number of sovereigns; and $27x + 20(x + 3) = 50 \times 20$; $47x = 940$; $x = 20$.

38. Let x be the number of shillings; then $6x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x + 30x = 1696$; $x = 53$.

39. Let x be the number of sovereigns; then $2x$ is the number of shillings, and $3x$ the number of pence; and, expressing all the quantities as pence, $240x + 24x + 3x = 1335$; $x = 5$.

XXXV.

1. $a^2 - b^2 = (a - b)(a + b)$; $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

2. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.

3. $a^2 - x^2 = (a + x)(a - x)$; $(a - x)^2 = (a - x)(a - x)$.

4. $a^3 + x^3 = (a + x)(a^2 - ax + x^2)$; $(a + x)^3 = (a + x)(a + x)(a + x)$.

5. $9x^2 - 1 = (3x + 1)(3x - 1)$; $(3x + 1)^2 = (3x + 1)(3x + 1)$.

6. $1 - 25a^2 = (1 + 5a)(1 - 5a)$; $(1 - 5a)^2 = (1 - 5a)(1 - 5a)$.

7. $x^2 - y^2 = (x + y)(x - y)$; $(x + y)^2 = (x + y)(x + y)$;
$x^2 + 3xy + 2y^2 = (x + y)(x + 2y)$.

8. $x^2 - y^2 = (x + y)(x - y)$; $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$;
$x^2 - 7xy + 6y^2 = (x - y)(x - 6y)$.

9. $x^2-1=(x-1)(x+1)$; $x^3-1=(x-1)(x^2+x+1)$;
$x^2+x-2=(x-1)(x+2)$.

10. $1-a^2=(1-a)(1+a)$; $1+a^3=(1+a)(1-a+a^2)$;
$a^2+5a+4=(1+a)(4+a)$.

XXXVI.

1. 6906)10359(1
6906
3453)6906(2
6906

∴ H. C. F. is 3453.

2. 1908)2736(1
1908
828)1908(2
1656
252)828(3
756
72)252(3
216
36)72(2
72

∴ H. C. F. is 36.

3. 49608)169416(3
148824
20592)49608(2
41184
8424)20592(2
16848
3744)8424(2
7488
936)3744(4
3744

∴ H. C. F. is 936.

4. 40115)126025(3
120345
5680)40115(7
39760
355)5680(16
355
2130
2130

∴ H. C. F. is 355.

5. 1581227)16758766(10
15812270
946496)1581227(1
946496
634731)946496(1
634731
311765)634731(2
623530
11201)311765(27
22402
87745
78407
9338)11201(1
9338
1863)9338(5
9315
23)1863(81
184
23
23

∴ H. C. F. is 23.

6. $35175)236845(6$

$\underline{211050}$

$25795)35175(1$

$\underline{25795}$

$9380)25795(2$

$\underline{18760}$

$7035)9380(1$

$\underline{7035}$

$2345)7035(3$

$\underline{7035}$

∴ H. C. F. is 2345.

XXXVII.

1. $x^2+7x+12)x^2+9x+20(1$

$\underline{x^2+7x+12}$

$2x+8$

Divide by 2; $x+4)x^2+7x+12(x+3$

$\underline{x^2+4x}$

$3x+12$

$\underline{3x+12}$

2. $x^2+12x+20)x^2+14x+40(1$

$\underline{x^2+12x+20}$

$2x+20$

Divide by 2; $x+10)x^2+12x+20(x+2$

$\underline{x^2+10x}$

$2x+20$

$\underline{2x+20}$

3. $x^2 - 13x + 42)\,x^2 - 17x + 70\,(1$

$\underline{x^2 - 13x + 42}$

$-4x + 28$

Divide by 4 and change signs ; $x - 7)\,x^2 - 13x + 42\,(x - 6$

$\underline{x^2 - 7x}$

$-6x + 42$

$\underline{-6x + 42}$

4. $x^2 + 5x - 84)\,x^2 + 21x + 108\,(1$

$\underline{x^2 + 5x - 84}$

$16x + 192$

Divide by 16 ; $x + 12)\,x^2 + 5x - 84\,(x - 7$

$\underline{x^2 + 12x}$

$-7x - 84$

$\underline{-7x - 84}$

5. $x^2 + x - 12)\,x^2 - 2x - 3\,(1$

$\underline{x^2 + x - 12}$

$-3x + 9$

Change signs and divide by 3 ; $x - 3)\,x^2 + x - 12\,(x + 4$

$\underline{x^2 - 3x}$

$4x - 12$

$\underline{4x - 12}$

6. $x^2 + 5xy + 6y^2)\,x^2 + 6xy + 9y^2\,(1$

$\underline{x^2 + 5xy + 6y^2}$

$xy + 3y^2$

Divide by y ; $x + 3y)\,x^2 + 5xy + 6y^2\,(x + 2y$

$\underline{x^2 + 3xy}$

$2xy + 6y^2$

$\underline{2xy + 6y^2}$

7. $x^2-6xy+8y^2)\,x^2-8xy+16y^2\,(1$

$$\underline{x^2-6xy+8y^2}$$

$$-2xy+8y^2$$

Divide by $2y$ and change signs ; $x-4y)\,x^2-6xy+8y^2\,(x-2y$

$$\underline{x^2-4xy}$$

$$-2xy+8y^2$$

$$\underline{-2xy+8y^2}$$

8. $x^2-13xy-30y^2)\,x^2-18xy+45y^2\,(1$

$$\underline{x^2-13xy-30y^2}$$

$$-5xy+75y^2$$

Divide by $5y$ and change signs ; $x-15y)\,x^2-13xy-30y^2\,(x+2y$

$$\underline{x^2-15xy}$$

$$2xy-30y^2$$

$$\underline{2xy-30y^2}$$

9. $x^2-2xy+y^2)\,x^3-y^3\,(x+2y$

$$\underline{x^3-2x^2y+xy^2}$$

$$2x^2y-xy^2-y^3$$

$$\underline{2x^2y-4xy^2+2y^3}$$

$$3xy^2-3y^3$$

Divide by $3y^2$; $x-y)\,x^2-2xy+y^2\,(x-y$.

10. $x^3+y^3)\,x^3+3x^2y+3xy^2+y^3\,(1$

$$\underline{x^3 \qquad\qquad +y^3}$$

$$3x^2y+3xy^2$$

Divide by $3xy$; $x+y)\,x^3+y^3\,(x^2-xy+y^2$.

11. $x^2-2xy+y^2)x^4-y^4(x^2+2xy+3y^2$

$$\underline{x^4-2x^3y+x^2y^2}$$

$$2x^3y-x^2y^2-y^4$$

$$\underline{2x^3y-4x^2y^2+2xy^3}$$

$$3x^2y^2-2xy^3-y^4$$

$$\underline{3x^2y^2-6xy^3+3y^4}$$

$$4xy^3-4y^4$$

Divide by $4y^3$; $x-y)x^2-2xy+y^2(x-y$.

12. $x^3+y^3)x^5+y^5(x^2$

$$\underline{x^5+x^2y^3}$$

$$-x^2y^3+y^5$$

Divide by $-y^3$; $x^2-y^2)x^3-y^3(x$

$$\underline{x^3-xy^2}$$

$$xy^2-y^3$$

Divide by y^2; $x-y)x^2-y^2(x+y$.

13. $x^2+2xy+y^2)x^4-y^4(x^2-2xy+3y^2$

$$\underline{x^4+2x^3y+x^2y^2}$$

$$-2x^3y-x^2y^2-y^4$$

$$\underline{-2x^3y-4x^2y^2-2xy^3}$$

$$3x^2y^2+2xy^3-y^4$$

$$\underline{3x^2y^2+6xy^3+3y^4}$$

$$-4xy^3-4y^4$$

Divide by $-4y^3$; $x+y)x^2+2xy+y^2(x+y$.

14. $a^2-b^2+2bc-c^2)a^2+2ab+b^2-2ac-2bc+c^2(1$

$$\underline{a^2-b^2+2bc-c^2}$$

$$2ab+2b^2-2ac-4bc+2c^2$$

Divide by 2; $ab+b^2-ac-2bc+c^2=(ab-ac)+(b^2-2bc+c^2)$
$=a(b-c)+(b-c)^2$.

Divide by $b-c$; $a+b-c)a^2-b^2+2bc-c^2(a-b+c$.

15. Multiply the second expression by 3;

$$12x^2+7xy+y^2)84x^2+9xy-3y^2(7$$
$$\underline{84x^2+49xy+7y^2}$$
$$-40xy-10y^2$$

Divide by $-10y$; $4x+y)12x^2+7xy+y^2(3x+y$.

16. Multiply the second expression by 2;

$$6x^2+xy-y^2)78x^2-44xy+6y^2(13$$
$$\underline{78x^2+13xy-13y^2}$$
$$-57xy+19y^2$$

Divide by $-19y$; $3x-y)6x^2+xy-y^2(2x+y$.

17. Multiply the second expression by 3;

$$15x^2-8xy+y^2)120x^2-9xy-3y^2(8$$
$$\underline{120x^2-64xy+8y^2}$$
$$55xy-11y^2$$

Divide by $11y$; $5x-y)15x^2-8xy+y^2(3x-y$.

18. $x^4+x^3-4x^2+x+1)x^5-5x^3+5x^2-1(x-1$

$$\underline{x^5+x^4-4x^3+x^2+x}$$
$$-x^4-x^3+4x^2-x-1$$
$$\underline{-x^4-x^3+4x^2-x-1}$$

19. $x^4+4x^2+16)x^5+x^4-2x^3+17x^2-10x+20(x+1$

$$\underline{x^5+4x^3+16x}$$
$$x^4-6x^3+17x^2-26x+20$$
$$\underline{x^4+4x^2+16}$$
$$-6x^3+13x^2-26x+4$$

Change the signs of the remainder, and multiply the divisor by 6;

$$6x^3-13x^2+26x-4)6x^4+24x^2+96(x$$
$$\underline{6x^4-13x^3+26x^2-4x}$$
$$13x^3-2x^2+4x+96$$

Multiply the remainder by 6, and continue the division ;

$$78x^3 - 12x^2 + 24x + 576(13$$
$$\underline{78x^3 - 169x^2 + 338x - 52}$$
$$157x^2 - 314x + 628$$

Divide by 157 ; $x^2 - 2x + 4)6x^3 - 13x^2 + 26x - 4(6x - 1$

$$\underline{6x^3 - 12x^2 + 24x}$$
$$-x^2 + 2x - 4$$
$$\underline{-x^2 + 2x - 4}$$

20. $x^4 + x^2y^2 + y^4)x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4(1$

$$\underline{x^4 \qquad + x^2y^2 \qquad + y^4}$$
$$2x^3y + 2x^2y^2 + 2xy^3$$

Divide by $2xy$; $x^2 + xy + y^2)x^4 + x^2y^2 + y^4(x^2 - xy + y^2$.

21. $x^6 - 6x^4 + 9x^2 - 4)x^6 + x^5 - 2x^4 + 3x^2 - x - 2(1$

$$\underline{x^6 \qquad - 6x^4 + 9x^2 \qquad - 4}$$
$$x^5 + 4x^4 - 6x^2 - x + 2$$

$$x^5 + 4x^4 - 6x^2 - x + 2)x^6 - 6x^4 + 9x^2 - 4(x - 4$$
$$\underline{x^6 + 4x^5 - 6x^3 - x^2 + 2x}$$
$$-4x^5 - 6x^4 + 6x^3 + 10x^2 - 2x - 4$$
$$\underline{-4x^5 - 16x^4 + 24x^2 + 4x - 8}$$
$$10x^4 + 6x^3 - 14x^2 - 6x + 4$$

Divide by 2, and multiply the divisor by 5 ;

$$5x^4 + 3x^3 - 7x^2 - 3x + 2)5x^5 + 20x^4 - 30x^2 - 5x + 10(x$$
$$\underline{5x^5 + 3x^4 - 7x^3 - 3x^2 + 2x}$$
$$17x^4 + 7x^3 - 27x^2 - 7x + 10$$

Multiply by 5, and continue the division ;

$$85x^4 + 35x^3 - 135x^2 - 35x + 50(17$$
$$\underline{85x^4 + 51x^3 - 119x^2 - 51x + 34}$$
$$-16x^3 - 16x^2 + 16x + 16$$

Divide by -16; $x^3+x^2-x-1)5x^4+3x^3-7x^2-3x+2(5x-2$

$$\underline{5x^4+5x^3-5x^2-5x}$$

$$-2x^3-2x^2+2x+2$$

$$\underline{-2x^3-2x^2+2x+2}$$

22. Multiply the first expression by 2;

$$6a^3+19a^2b+8ab^2-5b^3)30a^4+20a^3b+8a^2b^2+12ab^3-6b^4(5a$$

$$\underline{30a^4+95a^3b+40a^2b^2-25ab^3}$$

$$-75a^3b-32a^2b^2+37ab^3-6b^4$$

Change the signs, multiply by 2, divide by b, and continue the division;

$$6a^3+19a^2b+8ab^2-5b^3)150a^3+64a^2b-74ab^2+12b^3(25$$

$$\underline{150a^3+475a^2b+200ab^2-125b^3}$$

$$-411a^2b-274ab^2+137b^3$$

Divide by $-137b$; $3a^2+2ab-b^2)6a^3+19a^2b+8ab^2-5b^3(2a+5b.$

23. Multiply the second expression by 5;

$$15x^3-14x^2y+24xy^2-7y^3)135x^3+165x^2y-100xy^2+10y^3(9$$

$$\underline{135x^3-126x^2y+216xy^2-63y^3}$$

$$291x^2y-316xy^2+73y^3$$

Divide by y; and multiply the divisor by 97;

$$291x^2-316xy+73y^2)1455x^3-1358x^2y+2328xy^2-679y^3(5x$$

$$\underline{1455x^3-1580x^2y+365xy^2}$$

$$222x^2y+1963xy^2-679y^3$$

Divide by y; multiply by 97, and continue the division;

$$21534x^2+190411xy-65863y^2(74$$

$$\underline{21534x^2-\ 23384xy\ +\ \ 5402y^2}$$

$$213795xy-71265y^2$$

Divide by $71265y$; $3x-y)291x^2-316xy+73y^2(97x-73y.$

24. Multiply the second expression by 7;

$$21x^2-83xy-27x+22y^2+99y)\,84x^2-245xy-42x-231y^2+154y\,(4$$

$$\underline{84x^2-332xy-108x+88y^2+396y}$$

$$87xy+66x-319y^2-242y$$

$$\begin{aligned}\text{The remainder}&=87xy-319y^2+66x-242y\\&=29y(3x-11y)+22(3x-11y)\\&=(29y+22)(3x-11y)\end{aligned}$$

Rejecting $29y+22$, which is clearly not a factor of the divisor;

$$3x-11y)\,21x^2-83xy-27x+22y^2+99y\,(7x-2y-9.$$

25. $$\begin{aligned}3a^3-12a^2-a^2b+10ab-2b^2&=(3a^3-a^2b)-(12a^2-10ab+2b^2)\\&=a^2(3a-b)-2(6a^2-5ab+b^2)\\&=a^2(3a-b)-2(3a-b)(2a-b)\\&=(3a-b)(a^2-4a+2b)\end{aligned}$$

Rejecting $a^2-4a+2b$, which is clearly not a factor of the second expression, we find the H. C. F. to be $3a-b$.

26. $60a^2-75ax+15x^2=15(4a^2-5ax+x^2)=3\times5(a-x)(4a-x)$.

Of these four factors, 5 and $4a-x$ are clearly not factors of the first expression: but 3 is a factor of it: divide it by 3, and divide the result by $a-x$.

$$a-x)\,6a^3-6a^2x+2ax^2-2x^3\,(6a^2+2x^2$$

$\therefore$ H. C. F. is $3(a-x)$.

27. Divide the first expression by x, and multiply the result by 2;

$$6x^2-x-2)\,42x^2-52x+16\,(7$$

$$\underline{42x^2-7x-14}$$

$$-45x+30$$

Divide by -15; $3x-2)\,6x^2-x-2\,(2x+1$.

28. $$3x^3-15ax^2+a^2x-5a^3)\,6x^4+29a^2x^2+9a^4\,(2x+10a$$

$$\underline{6x^4-30ax^3+2a^2x^2-10a^3x}$$

$$30ax^3+27a^2x^2+10a^3x+9a^4$$

$$\underline{30ax^3-150a^2x^2+10a^3x-50a^4}$$

$$177a^2x^2\qquad+59a^4$$

Divide by $59a^2$; $3x^2+a^2)\,3x^3-15ax^2+a^2x-5a^3\,(x-5a$.

29. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$

$$x^8 + x^6y^2 + x^2y + y^3 = x^6(x^2 + y^2) + y(x^2 + y^2)$$

$\therefore x^2 + y^2$ is the H. C. F.

30. Divide the first expression by 2;

$$x^3 + x^2 + 7x + 39)x^3 + 5x^2 + 7x + 3(1$$
$$\underline{x^3 + x^2 + 7x + 39}$$
$$4x^2 - 36$$

Now $4x^2 - 36 = 4(x^2 - 9) = 4(x + 3)(x - 3)$; and rejecting 4 and $x - 3$, we have

$$x + 3)x^3 + x^2 + 7x + 39(x^2 - 2x + 13.$$

31. $45a^3x + 3a^2x^2 - 9ax^3 + 6x^4 = x(45a^3 + 3a^2x - 9ax^2 + 6x^3)$

$18a^2x - 8x^3 = 2x(9a^2 - 4x^2)$

Reserving the common factor x;

$$9a^2 - 4x^2)45a^3 + 3a^2x - 9ax^2 + 6x^3(5a$$
$$\underline{45a^3 - 20ax^2}$$
$$3a^2x + 11ax^2 + 6x^3$$

Divide by x, multiply by 3, and proceed with the division;

$$9a^2 + 33ax + 18x^2(1$$
$$\underline{9a^2 - 4x^2}$$
$$33ax + 22x^2$$

Divide by $11x$; $3a + 2x)9a^2 - 4x^2(3a - 2x$

$\therefore$ the H. C. F. is $(3a + 2x)x$.

XXXVIII.

1. $x^2 + 5x + 6 = (x + 2)(x + 3)$; $x^2 + 7x + 10 = (x + 2)(x + 5)$; $x^2 + 12x + 20 = (x + 2)(x + 10)$.

2. $x^3 + 4x^2 - 5)x^3 - 3x + 2(1$

$$\underline{x^3 + 4x^2 - 5}$$
$$-4x^2 - 3x + 7$$

Change signs and multiply the divisor by 4;

$$4x^2+3x-7)\,4x^3+16x^2-20\,(x$$
$$\underline{4x^3+3x^2-7x}$$
$$13x^2+7x-20$$

Multiply by 4, and continue the division;

$$52x^2+28x-80\,(13$$
$$\underline{52x^2+39x-91}$$
$$-11x+11$$

Change signs and divide by 11;

$$x-1)\,4x^2+3x-7\,(4x+7$$

Hence $x-1$ is the H. C. F. of the first two expressions; and since $x-1$ divides x^3+4x^2-8x+3 exactly, it is the H. C. F. required.

3. $2x^2+x-1=(2x-1)(x+1)$; $x^2+5x+4=(x+4)(x+1)$;
$x^3+1=(x+1)(x^2-x+1)$.

4. $y^3-y^2-y+1=(y^3-y^2)-(y-1)=y^2(y-1)-(y-1)=(y^2-1)(y-1)$
$3y^2-2y-1=(3y+1)(y-1)$
$y^3-y^2+y-1=y^2(y-1)+(y-1)=(y^2+1)(y-1)$.

5. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$\underline{x^3-4x^2+9x-10}$$
$$6x^2-12x+30$$

Divide by 6; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

$$\underline{x^3-2x^2+5x}$$
$$-2x^2+4x-10$$
$$\underline{-2x^2+4x-10}$$

Hence x^2-2x+5 is the H. C. F. of the first two expressions, and as it also divides $x^3+5x^2-9x+35$ exactly, it is the H. C. F. required.

6. $x^3-7x^2+16x-12)\,3x^3-14x^2+16x\,(3$
$3x^3-21x^2+48x-36$
$7x^2-32x+36$

Multiply the divisor by 7;
$7x^2-32x+36)\,7x^3-49x^2+112x-84\,(x$
$7x^3-32x^2+36x$
$-17x^2+76x-84$

Multiply by -7, and continue the division;
$119x^2-532x+588\,(17$
$119x^2-544x+612$
$12x-24$

Divide by 12; $x-2)\,7x^2-32x+36\,(7x-18$
Hence $x-2$, which also divides $5x^3-10x^2+7x-14$, is the H. C. F.

7. $y^3-5y^2+11y-15)\,y^3-y^2+3y+5\,(1$
$y^3-5y^2+11y-15$
$4y^2-8y+20$

Divide by 4; $y^2-2y+5)\,y^3-5y^2+11y-15\,(y-3$
y^3-2y^2+5y
$-3y^2+6y-15$
$-3y^2+6y-15$

Hence as y^2-2y+5 divides $2y^3-7y^2+16y-15$ exactly, it is the H. C. F.

XXXIX.

11. $$\frac{a^2}{a^2+ab}=\frac{a^2}{a(a+b)}=\frac{a}{a+b}.$$

12. $$\frac{14m^2x}{21m^3p-7mx}=\frac{14m^2x}{7m(3m^2p-x)}=\frac{2mx}{3m^2p-x}.$$

13. $$\frac{xy}{3xy^2-5x^2yz}=\frac{xy}{xy(3y-5xz)}=\frac{1}{3y-5xz}.$$

14. $\dfrac{4ax+2x^2}{8ax^3-2x^2}=\dfrac{2x(2a+x)}{2x(4ax^2-x)}=\dfrac{2a+x}{4ax^2-x}.$

15. $\dfrac{ay+y^2}{abc+bcy}=\dfrac{y(a+y)}{bc(a+y)}=\dfrac{y}{bc}.$

16. $\dfrac{4a^2x+6a^2y}{8x^2-18y^2}=\dfrac{2a^2(2x+3y)}{2(2x+3y)\,(2x-3y)}=\dfrac{a^2}{2x-3y}.$

17. $\dfrac{12ab^2-6ab}{8b^2c-2c}=\dfrac{6ab(2b-1)}{2c(2b+1)\,(2b-1)}=\dfrac{3ab}{2bc+c}.$

18. $\dfrac{c^2-4a^2}{c^2+4ac+4a^2}=\dfrac{(c+2a)\,(c-2a)}{(c+2a)\,(c+2a)}=\dfrac{c-2a}{c+2a}.$

19. $\dfrac{3x^4+3x^2y^2}{5x^4+5x^2y^2}=\dfrac{3x^2(x^2+y^2)}{5x^2(x^2+y^2)}=\dfrac{3}{5}.$

20. $\dfrac{10x-10y}{4x^2-8xy+4y^2}=\dfrac{10(x-y)}{4(x-y)\,(x-y)}=\dfrac{5}{2x-2y}.$

21. $\dfrac{ax+by}{7a^2x^2-7b^2y^2}=\dfrac{ax+by}{7(ax+by)\,(ax-by)}=\dfrac{1}{7ax-7by}.$

22. $\dfrac{6ab+8cd}{27a^2b^2x-48c^2d^2x}=\dfrac{2(3ab+4cd)}{3x(3ab+4cd)(3ab-4cd)}=\dfrac{2}{9abx-12cdx}.$

23. $\dfrac{xy-xyz}{2az-2az^2}=\dfrac{xy(1-z)}{2az(1-z)}=\dfrac{xy}{2az}.$

24. $\dfrac{7ab^3x^8-7ab^3y^2}{14a^3bcx^8-14a^3bcy^2}=\dfrac{7ab^3(x^8-y^2)}{14a^3bc(x^8-y^2)}=\dfrac{b^2}{2a^2c}.$

25. $\dfrac{5x^9+45dx^2}{10cx^9+90cdx^2}=\dfrac{5x^2(x^7+9d)}{10cx^2(x^7+9d)}=\dfrac{1}{2c}.$

26. $\dfrac{10a^2+20ab+10b^2}{5a^3+5a^2b}=\dfrac{10(a+b)\,(a+b)}{5a^2(a+b)}=\dfrac{2a+2b}{a^2}.$

27. $\dfrac{4x^2-8xy+4y^2}{48(x-y)^2}=\dfrac{4(x-y)^2}{48(x-y)^2}=\dfrac{1}{12}.$

28. $\dfrac{3mx+5nx^2}{3my+5nxy}=\dfrac{x(3m+5nx)}{y(3m+5nx)}=\dfrac{x}{y}.$

XL.

1. $\dfrac{a^2+7a+10}{a^2+5a+6}=\dfrac{(a+5)(a+2)}{(a+3)(a+2)}=\dfrac{a+5}{a+3}.$

2. $\dfrac{x^2-9x+20}{x^2-7x+12}=\dfrac{(x-4)(x-5)}{(x-4)(x-3)}=\dfrac{x-5}{x-3}.$

3. $\dfrac{x^2-2x-3}{x^2-10x+21}=\dfrac{(x-3)(x+1)}{(x-3)(x-7)}=\dfrac{x+1}{x-7}.$

4. $\dfrac{x^2-18xy+45y^2}{x^2-8xy-105y^2}=\dfrac{(x-3y)(x-15y)}{(x+7y)(x-15y)}=\dfrac{x-3y}{x+7y}.$

5. $\dfrac{x^4+x^2+1}{x^2+x+1}=\dfrac{(x^2+x+1)(x^2-x+1)}{x^2+x+1}=x^2-x+1.$

6. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}=\dfrac{(x^3+y^3)(x^3+y^3)}{(x^3+y^3)(x^3-y^3)}=\dfrac{x^3+y^3}{x^3-y^3}.$

7. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$\begin{array}{r} x^3-4x^2+9x-10 \\ \hline 6x^2-12x+30 \end{array}$$

Divide by 6 ; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

Hence x^2-2x+5 is the H. C. F.

8. $x^3-5x^2+11x-15)\,x^3-x^2+3x+5\,(1$

$$\begin{array}{r} x^3-5x^2+11x-15 \\ \hline 4x^2-8x+20 \end{array}$$

Divide by 4 ; $x^2-2x+5)\,x^3-5x^2+11x-15\,(x-3$

Hence x^2-2x+5 is the H. C. F.

9. $x^3-8x^2+21x-18)\,3x^3-16x^2+21x\,(3$

$$\begin{array}{r} 3x^3-24x^2+63x-54 \\ \hline 8x^2-42x+54 \end{array}$$

Divide by 2 ; $4x^2-21x+27$

Multiply divisor by 4; $4x^2-21x+27)\,4x^3-32x^2+84x-72\,(x$

$4x^3-21x^2+27x$

$-11x^2+57x-72$

Multiply by -4, and continue the division;

$4x^2-21x+27)\,44x^2-228x+288\,(11$

$44x^2-231x+297$

$3x-9$

Divide by 3; $x-3)\,4x^2-21x+27\,(4x-9$

Hence $x-3$ is the H. C. F.

10. The H. C. F. is $x-2$, see the work in xxxviii. 6.

11. $x^4-x^3y-xy^3-y^4)\,x^4+x^3y+xy^3-y^4\,(1$

$x^4-x^3y-xy^3-y^4$

$2x^3y+2xy^3$

Divide by $2xy$; $x^2+y^2)\,x^4-x^3y-xy^3-y^4\,(x^2-xy-y^2$.

12. $a^3-3a+2)\,a^3+4a^2-5\,(1$

a^3-3a+2

$4a^2+3a-7$

Multiply divisor by 4; $4a^2+3a-7)\,4a^3-12a+8\,(a$

$4a^3+3a^2-7a$

$-3a^2-5a+8$

Multiply by -4 and continue the division;

$12a^2+20a-32\,(3$

$12a^2+9a-21$

$11a-11$

Divide by 11; $a-1)\,4a^2+3a-7\,(4a+7$.

13. $b^3-6b+5)\,b^3+4b^2-5b\,(1$

b^3-6b+5

$4b^2+b-5$

Multiply divisor by 4 ; $4b^2+b-5)\,4b^3-24b+20\,(b$

$4b^3+b^2-5b$

$-b^2-19b+20$

Change signs ; $b^2+19b-20)\,4b^2+b-5\,(4$

$4b^2+76b-80$

$-75b+75$

Hence the H. C. F. is found to be $b-1$.

14. $m^3-7m+6)\,m^3+3m^2-4m\,(1$

m^3-7m+6

$3m^2+3m-6$

Divide by 3 ; $m^2+m-2)\,m^3-7m+6\,(m-1$

m^3+m^2-2m

$-m^2-5m+6$

$-m^2-m+2$

$-4m+4$

Hence the H. C. F. is found to be $m-1$.

15. $a^3+1)\,a^3+2a^2+2a+1\,(1$

$a^3 \qquad +1$

$2a^2+2a$

Divide by $2a$, and H. C. F. is found to be $a+1$.

16. $$\frac{3ax^2-13ax+14a}{7x^3-17x^2+6x}=\frac{a(3x^2-13x+14)}{x(7x^2-17x+6)}$$

$3x^2-13x+14)\,21x^2-51x+18\,(7$

$21x^2-91x+98$

$40x-80$

Whence H. C. F. of $3x^2-13x+14$ and $7x^2-17x+6$ is found to be $x-2$.

17. $$\frac{14x^2-34x+12}{9ax^2-39ax+42a}=\frac{2(7x^2-17x+6)}{3a(3x^2-13x+14)};$$

and as in the preceding question, the H. C. F. is found to be $x-2$.

18. $\frac{10a-24a^2+14a^3}{15-24a+3a^2+6a^3}=\frac{2a(5-12a+7a^2)}{3(5-8a+a^2+2a^3)}$

$$5-12a+7a^2)\,5-8a+a^2+2a^3\,(1$$
$$\underline{5-12a+7a^2}$$
$$4a-6a^2+2a^3$$

Divide by $2a$; $2-3a+a^2=(1-a)(1-2a)$

Hence the H. C. F. is found to be $1-a$.

19. $\frac{2ab^3+ab^2-8ab+5a}{7b^3-12b^2+5b}=\frac{a(2b^3+b^2-8b+5)}{b(7b^2-12b+5)}$

$$7b^2-12b+5)\,14b^3+7b^2-56b+35\,(2b$$
$$\underline{14b^3-24b^2+10b}$$
$$31b^2-66b+35$$
$$\underline{7}$$
$$217b^2-462b+245\,(31$$
$$\underline{217b^2-372b+155}$$
$$-90b+90$$

Hence the H. C. F. is found to be $b-1$.

20. $a^3-4a^2+6a-4)\,a^3-3a^2+3a-2\,(1$

$$\underline{a^3-4a^2+6a-4}$$
$$a^2-3a+2)\,a^3-4a^2+6a-4\,(a-1$$
$$\underline{a^3-3a^2+2a}$$
$$-a^2+4a-4$$
$$\underline{-a^2+3a-2}$$
$$a-2$$

Hence we find $a-2$ to be the H. C. F.

21. $\frac{3x^2+2x-1}{x^3+x^2-x-1}=\frac{(3x-1)(x+1)}{(x^2-1)(x+1)}=\frac{3x-1}{x^2-1}.$

22. $\frac{a^2-a-20}{a^2+a-12}=\frac{(a-5)(a+4)}{(a-3)(a+4)}=\frac{a-5}{a-3}.$

23. $$\begin{array}{l} x^3-3x^2+4x-2)\,x^3-x^2-2x+2\,(1 \\ \qquad\qquad\qquad\quad \underline{x^3-3x^2+4x-2} \\ \qquad\qquad\qquad\qquad\quad 2x^2-6x+4 \end{array}$$

Divide by 2; $$\begin{array}{l} x^2-3x+2)\,x^3-3x^2+4x-2\,(x \\ \qquad\qquad\quad \underline{x^3-3x^2+2x} \\ \qquad\qquad\qquad\qquad\quad 2x-2 \end{array}$$

Hence we find $x-1$ to be the H. C. F.

24. $$\frac{x^2+y^2+z^2+2xy+2xz+2yz+z^2-2yz+y^2+x^2-2xz+z^2+y^2-2xy+x^2}{x^2+y^2+z^2}$$

$$=\frac{3x^2+3y^2+3z^2}{x^2+y^2+z^2}=3.$$

25. Multiply the numerator by 7;

$$\begin{array}{l} 7x^3-19x^2+17x-5)\,14x^4-7x^3-63x^2+91x-35\,(2x \\ \qquad\qquad\qquad\qquad\quad \underline{14x^3-38x^3+34x^2-10x} \\ \qquad\qquad\qquad\qquad\qquad\quad 31x^3-97x^2+101x-35 \\ \qquad\qquad\qquad\qquad\qquad\quad \underline{\;7\qquad\qquad\qquad\qquad\qquad} \\ \qquad\qquad\qquad\qquad\qquad\; 217x^3-679x^2+707x-245\,(31 \\ \qquad\qquad\qquad\qquad\qquad\; \underline{217x^3-589x^2+527x-155} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad -90x^2+180x-90 \end{array}$$

Divide by -90; $$\begin{array}{l} x^2-2x+1)\,7x^3-19x^2+17x-5\,(7x-5 \\ \qquad\qquad\quad \underline{7x^3-14x^2+7x} \\ \qquad\qquad\qquad\quad -5x^2+10x-5 \\ \qquad\qquad\qquad\quad \underline{-5x^2+10x-5} \end{array}$$

Hence x^2-2x+1 is the H. C. F.

26. $$\begin{array}{l} 8x^4-30x^3+31x^2-12)\,16x^4-53x^2+45x+6\,(2 \\ \qquad\qquad\qquad\qquad\quad \underline{16x^4-60x^3+62x^2-24} \\ \qquad\qquad\qquad\qquad\qquad\quad 60x^3-115x^2+45x+30 \end{array}$$

Divide by 5, and multiply the divisor by 3;

$$12x^3-23x^2+9x+6)\,24x^4-90x^3+93x^2-36\,(2x$$
$$\underline{12x^4-46x^3+18x^2+12x}$$
$$-44x^3+75x^2-12x-36$$

Multiply by -3, and continue the division;

$$132x^3-225x^2+36x+108\,(11$$
$$\underline{132x^3-253x^2+99x+\ 66}$$
$$28x^2-63x+\ 42$$

Divide by 7; $4x^2-9x+6)\,12x^3-23x^2+9x+6\,(3x+1$

Hence $4x^2-9x+6$ is the H. C. F.

27. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}=\dfrac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}.$

28. $6x^3-23x^2+16x-3)\,6x^3-17x^2+11x-2\,(1$
$$\underline{6x^3-23x^2+16x-3}$$
$$6x^2-\ 5x+1$$

$6x^2-5x+1)\,6x^3-23x^2+16x-3\,(x-3$

Hence $6x^2-5x+1$ is the H. C. F.

29. $x^3-6x^2+11x-6)\,x^3-2x^2-\ x\ +2\,(1$
$$\underline{x^3-6x^2+11x-6}$$
$$4x^2-12x+8$$

Divide by 4; $x^2-3x+2)\,x^3-6x^2+11x-6\,(x-3$

Hence x^2-3x+2 is the H. C. F.

30. $m^3+m^2+m-3)\,m^3+3m^2+5m+3\,(1$
$$\underline{m^3+\ m^2+\ m-3}$$
$$2m^2+4m+6$$

Divide by 2; $m^2+2m+3)\,m^3+m^2+m-3\,(m-1$

Hence m^2+2m+3 is the H. C. F.

31. $\dfrac{x^5+5x^4-x^2-5x}{x^4+3x^3-x-3}=\dfrac{x^4(x+5)-x(x+5)}{x^3(x+3)-(x+3)}=\dfrac{(x+5)(x^3-1)x}{(x+3)(x^3-1)}=\dfrac{x^2+5x}{x+3}.$

32. $$\frac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}=\frac{a^2-(b^2+2bc+c^2)}{(a^2+2ab+b^2)-c^2}=\frac{a^2-(b+c)^2}{(a+b)^2-c^2}$$
$$=\frac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}=\frac{a-b-c}{a+b-c}.$$

33. Multiply the numerator by 3;
$$\begin{array}{r} 9a^2+3ab-2b^2)\,45a^2+\ \ 3ab-\ \ 6b^2\,(5 \\ \underline{45a^2+15ab-10b^2} \\ -12ab+\ \ 4b^2 \end{array}$$
Divide by $-4b$; $3a-b)\,9a^2+3ab-2b^2\,(3a+2b$
Hence $3a-b$ is the H. C. F.

34. $x^2-7x+10=(x-5)(x-2)$
$x-5$ is clearly not a factor of the denominator;
$$x-2)\,2x^2-x-6\,(2x+3$$
Hence $x-2$ is the H. C. F.

35. $$\begin{array}{r} x^3+3x^2+4x+12)\,x^3+4x^2+4x+3\,(1 \\ \underline{x^3+3x^2+4x+12} \\ x^2-9 \end{array}$$
Divide by $x-3$; $x+3)\,x^3+3x^2+4x+12\,(x^2+4$
Hence $x+3$ is the H. C. F.

36. $$\begin{array}{r} 2x^3-x-1)\,2x^4-2x^2-4x+4\,)x \\ \underline{2x^4-\ \ x^2-x} \\ -\ \ x^2-3x+4 \end{array}$$
Change signs; $x^2+3x-4=(x+4)(x-1)$
Divide by $x+4$; $x-1)\,2x^3-x-1\,(2x^2+2x+1$
Hence $x-1$ is the H. C. F.

37. $$\begin{array}{r} 3x^2-4x-15)\,3x^3-6x^2-45x+108\,(x \\ \underline{3x^3-4x^2-15x} \\ -2x^2-30x+108 \end{array}$$
Divide by -2; $x^2+15x-54=(x+18)(x-3)$
Divide by $x+18$; $x-3)\,3x^2-4x-15\,(3x+5$
Hence $x-3$ is the H. C. F.

38. $3x^3+x^2-5x+21)\,6x^3+29x^2+26x-21\,(2$

$\underline{6x^3+\ \ 2x^2-10x+42}$

$27x^2+36x-63$

Divide by 9; $3x^2+4x-7)\,3x^3+\ x^2-5x+21\,(x-1$

$\underline{3x^3+4x^2-7x}$

$-3x^2+2x+21$

$\underline{-3x^2-4x+\ \ 7}$

$6x+14$

Divide by 2; $3x+7)\,3x^2+4x-7\,(x-1$

Hence $3x+7$ is the H. C. F.

39. $4x^3-3x^2-8x-1)\,4x^4-4x^3-16x^2-4x+4\,(x$

$\underline{4x^4-3x^3-\ \ 8x^2-x}$

$-\ \ x^3-\ \ 8x^2-3x+4$

Change signs; $x^3+8x^2+3x-4)\,4x^3-\ \ 3x^2-\ \ 8x-\ \ 1\,(4$

$\underline{4x^3+32x^2+12x-16}$

$-35x^2-20x+15$

Divide by -5; $7x^2+4x-3=(7x-3)(x+1)$

Divide by $7x-3$; $x+1)\,x^3+8x^2+3x-4\,(x^2+7x-4$

Hence $x+1$ is the H. C. F.

40. $a^3-7a^2+16a-12)\,3a^3-14a^2+16a\,(3$

$\underline{3a^3-21a^2+48a-36}$

$7a^2-32a+36$

$7a^2-32a+36)\,7a^3-49a^2+112a-84\,(a$

$\underline{7a^3-32a^2+\ \ 36a}$

$-17a^2+\ \ 76a-84$

$\underline{7}$

$-119a^2+532a-588\,(-17$

$\underline{-119a^2+544a-612}$

$-12a+24$

Hence we find $a-2$ to be the H. C. F.

XLII.

1. $\frac{a-b}{a^2+ab}\times\frac{a^2-b^2}{a^2-ab}=\frac{(a-b)(a+b)(a-b)}{a(a+b)a(a-b)}=$ etc.

2. $\frac{x^2+4x}{x^2-3x}\times\frac{4x^2-12x}{3x^2+12x}=\frac{x(x+4)4x(x-3)}{x(x-3)3x(x+4)}=$ etc.

3. $\frac{x^2+3x+2}{x^2-5x+6}\times\frac{x^2-7x+12}{x^2+x}=\frac{(x+2)(x+1)(x-3)(x-4)}{(x-2)(x-3)x(x+1)}=$ etc.

4. $\frac{x^2+x-2}{x^2-7x}\times\frac{x^2-13x+42}{x^2+2x}=\frac{(x+2)(x-1)(x-7)(x-6)}{x(x-7)x(x+2)}=$ etc.

5. $\frac{x^2-11x+30}{x^2-6x+9}\times\frac{x^2-3x}{x^2-5x}=\frac{(x-6)(x-5)x(x-3)}{(x-3)(x-3)x(x-5)}=$ etc.

6. $\frac{x^2-4}{x^2+5x}\times\frac{x^2-25}{x^2+2x}=\frac{(x+2)(x-2)(x+5)(x-5)}{x(x+5)x(x+2)}=$ etc.

7. $\frac{(a-3)(a-1)}{(a-4)(a-1)}\times\frac{(a-5)(a-4)}{(a-7)(a-3)}\times\frac{a(a-7)}{a(a-5)}=1.$

8. $\frac{(b-6)(b-1)}{(b+4)(b-1)}\times\frac{(b+4)(b+6)}{(b-8)(b-6)}\times\frac{b^2(b-8)}{b(b+6)}=b.$

9. $\frac{(x+y)(x-y)}{(x-y)(x-2y)}\times\frac{y(x-2y)}{x(x+y)}\times\frac{x(x-y)}{(x-y)(x-y)}=\frac{y}{x-y}.$

10. $\frac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)}\times\frac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)}=\frac{c-a+b}{c-a-b}.$

11. $\frac{(x-m+n)(x-m-n)}{(x-n+m)(x-n-m)}\times\frac{(x+n-m)(x-n+m)}{(x+m-n)(x-m+n)}=\frac{x-m+n}{x+m-n}.$

12. $\frac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)}\times\frac{(a-b+d-c)(a-b-d+c)}{(a-c+d-b)(a-c-d+b)}=1.$

13. $\frac{(x^2-2xy+y^2)-z^2}{(x^2+2xy+y^2)-z^2}\times\frac{x+y-z}{x-y+z}=\frac{(x-y+z)(x-y-z)(x+y-z)}{(x+y+z)(x+y-z)(x-y+z)}=$ etc.

XLIII.

1. $\frac{2a}{x}\times\frac{5c}{3b}=\frac{10ac}{3bx}.$

2. $\frac{15y}{14z}\times\frac{7z}{5y^2}=\frac{3}{2y}.$

3. $\frac{8x^4y}{15ab^3}\times\frac{30ab^2}{2x^3}=\frac{8xy}{b}.$

4. $\frac{4a}{nx}\times\frac{1}{3ab}=\frac{4}{3bnx}.$

5. $\frac{3p}{2(p-1)} \times \frac{p-1}{2p} = \frac{3}{4}$.

6. $1 \times \frac{5x}{4a} = \frac{5x}{4a}$.

7. $\frac{5x}{7} \times \frac{1}{2} = \frac{5x}{14}$.

8. $\frac{1}{(x-1)(x-2)} \times \frac{x-1}{1} = \frac{1}{x-2}$.

9. $\frac{1}{(x-15)(x-2)} \times \frac{x-15}{1} = \frac{1}{x-2}$.

XLV.

1. $x^2 = x \times x$; $ax + x^2 = x(a+x)$; $\therefore$ L. C. M. is $x \times x(a+x)$.

2. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $\therefore$ L. C. M. is $(x+1)(x-1)x$.

3. $a^2 - b^2 = (a+b)(a-b)$; $a^2 + ab = a(a+b)$; $\therefore$ L. C. M. is $(a+b)(a-b)a$.

4. $4x^2 - 1 = (2x-1)(2x+1)$; $\therefore$ L. C. M. is $4x^2 - 1$.

5. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$; $\therefore$ L. C. M. is $a^3 + b^3$.

6. $x^2 - 1 = (x+1)(x-1)$; $\therefore$ L. C. M. is $x^2 - 1$.

7. $x^3 - 1 = (x-1)(x^2 + x + 1)$; $\therefore$ L. C. M. is $(x+1)(x^3 - 1)$.

8. $x^3 + 1 = (x+1)(x^2 - x + 1)$; $\therefore$ L. C. M. is $(x^2+1)(x^3+1)$.

9. $x^2 - 1 = (x+1)(x-1)$; $x^3 - 1 = (x-1)(x^2 + x + 1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)(x^2+x+1)$.

10. $x^4 - 1 = (x^2+1)(x^2-1)$; $\therefore$ L. C. M. is $x^4 - 1$.

11. $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$; $x^3 + 1 = (x+1)(x^2 - x + 1)$;
$\therefore$ L. C. M. is $x(x-1)(x^2+x+1)(x+1)(x^2-x+1)$, etc.

12. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)x(x^2+x+1)$, etc.

13. $4a^2 - 1 = (2a+1)(2a-1)$; $8a^3 + 1 = (2a+1)(4a^2 - 2a + 1)$;
$\therefore$ L. C. M. is $(2a+1)(2a-1)(4a^2 - 2a + 1)$, etc.

14. $2x^2+2xy=2x(x+y)$; $\therefore$ L. C. M. is $2x^2+2xy$.

15. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is $(a+b)(a+b)(a-b)$.

16. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is a^2-b^2.

17. $2(1-x^2)=2(1+x)(1-x)$; $\therefore$ L. C. M. is $4(1+x)(1-x)$.

18. $x^3-1=(x-1)(x^2+x+1)$; $\therefore$ L. C. M. is x^3-1.

19. L. C. M. is $(a-b)(a-c)(b-c)$.

20. L. C. M. is $(x+1)(x+2)(x+3)$.

21. $x^2-y^2=(x+y)(x-y)$;
$\therefore$ L. C. M. is $(x+y)(x-y)(x+y)(x-y)$.

22. $a^2-1=(a+1)(a-1)$; $\therefore$ L. C. M. is $(a+3)(a+1)(a-1)$.

23. $x(x^2-y^2)=x(x+y)(x-y)$; $\therefore$ L. C. M. is $x^2(x-y)(x+y)$.

24. L. C. M. is $(x+1)(x+3)(x+2)(x+4)$.

25. $x^2-y^2=(x+y)(x-y)$; $12(x^3+y^3)=12(x+y)(x^2-xy+y^2)$;
$\therefore$ L. C. M. is $12(x+y)(x-y)(x-y)(x^2-xy+y^2)$.

26. $6(x^2+xy)=6x(x+y)$; $8(xy-y^2)=8y(x-y)$;
$10(x^2-y^2)=10(x+y)(x-y)$;
$\therefore$ L. C. M. is $120xy(x+y)(x-y)$.

XLVI.

1. $x^2+5x+6=(x+2)(x+3)$
$x^2+6x+8=(x+2)(x+4)$.

2. $a^2-a-20=(a-5)(a+4)$
$a^2+a-12=(a-3)(a+4)$.

3. $x^2+3x+2=(x+1)(x+2)$
$x^2+4x+3=(x+1)(x+3)$.

4. $x^2+11x+30=(x+5)(x+6)$
$x^2+12x+35=(x+5)(x+7)$.

5. $x^2-\ 9x-22=(x-11)(x+2)$
$x^2-13x+22=(x-11)(x-2)$.

6. $2x^2+3x+1=(2x+1)(x+1)$
$x^2-x-2=(x-2)(x+1)$.

7. $x^3+x^2y+xy+y^2=(x^2+y)(x+y)$
$x^4-y^4=(x^2+y^2)(x+y)(x-y)$.

8. $x^2-8x+15=(x-3)(x-5)$
$x^2+2x-15=(x-3)(x+5)$.

9. $21x^2-26x+8)21x^3-12x^2-63x+36(x$
$\underline{21x^3-26x^2+8x}$
$14x^2-71x+36$
$\underline{3}$
$42x^2-213x+108(2$
$\underline{42x^2-52x+16}$
$-161x+92$
Divide by -23; $7x-4)21x^2-26x+8(3x-2$
$\therefore$ L. C. M. is $(3x-2)(7x^3-4x^2-21x+12)$
or $(3x-2)(7x-4)(x^2-3)$.

10. $x^3+x^2y+xy^2+y^3=(x^2+y^2)(x+y)$
$x^3-x^2y+xy^2-y^3=(x^2+y^2)(x-y)$.

11. $a^3-2a^2b-ab^2+2b^3)a^3+2a^2b-ab^2-2b^3(1$
$\underline{a^3-2a^2b-ab^2+2b^3}$
$4a^2b\quad -4b^3$
Divide by $4b$; $a^2-b^2)a^3-2a^2b-ab^2+2b^3(a-2b$
$\therefore$ L. C. M. is $(a-2b)(a^3+2a^2b-ab^2-2b^3)$
or $(a-2b)(a^2-b^2)(a+2b)$.

XLVII.

1. $x^2-3x+2=(x-1)(x-2)$
$x^2-4x+3=(x-1)(x-3)$
$x^2-5x+4=(x-1)(x-4)$.

2. $x^2+5x+4=(x+1)(x+4)$
$x^2+4x+3=(x+1)(x+3)$
$x^2+7x+12=(x+3)(x+4)$.

3. $x^2-\ \ 9x+20=(x-4)(x-5)$
$x^2-12x+35=(x-7)(x-5)$
$x^2-11x+28=(x-4)(x-7)$.

4. $$\begin{array}{r} 6x^2-x-2)\,42x^2-34x+\ \ 4\,(7 \\ \underline{42x^2-\ \ 7x-14} \\ -27x+18 \end{array}$$

Divide by -9; $3x-2)\,6x^2-x-2\,(2x+1$.

Hence L. C. M. of first two expressions is

$(2x+1)(21x^2-17x+2)$, or $42x^3-13x^2-13x+2$.

We have now to find the L. C. M. of this and $14x^2+5x-1$.

$$\begin{array}{r} 14x^2+5x-1)\,42x^3-13x^2-13x+2\,(3x-2 \\ \underline{42x^3+15x^2-\ \ 3x\qquad} \\ -28x^2-10x+2 \\ \underline{-28x^2-10x+2} \end{array}$$

$\therefore$ L. C. M. is $(3x-2)(14x^2+5x-1)$, etc.

5. $x^2-1=(x+1)(x-1)$
$x^2+2x-3=(x+3)(x-1)$
$6x^2-x-2=(3x-2)(2x+1)$.

6. $x^3-27=(x-3)(x^2+3x+9)$
$x^2-15x+36=(x-3)(x-12)$
$x^3-3x^2-2x+6=x^2(x-3)-2(x-3)=(x^2-2)(x-3)$.

XLVIII.

1. L. C. D. 20; new numerators $15x$, $16x$.
2. L. C. D. 18; new numerators $9x-21$, $4x-9$.
3. L. C. D. $10x^2$; new numerators $4x-8y$, $3x^2-8xy$.
4. L. C. D. $10a^2$; new numerators $20a+25b$, $6a^2-8ab$.
5. L. C. D. $60a^2c$; new numerators $48a^2-60ac$, $15a-10c$.
6. L. C. D. a^3b^2; new numerators $ab-b^2$, a^4-a^3b.
7. L. C. D. $1-x^2$; new numerators $3-3x$, $3+3x$.
8. L. C. D. $1-y^4$; new numerators $2+2y^2$, $2-2y^2$.
9. L. C. D. $1-x^2$; new numerators $5+5x$, 6.
10. L. C. D. $c(b+x)$; new numerators $ab+ax$, b.
11. L. C. D. $(a-b)(b-c)(a-c)$; new numerators $a-c$, $b-c$.
12. L. C. D. $abc(a-b)(a-c)(b-c)$; new numerators $bc-c^2$, $ab-b^2$.

XLIX.

1. $\frac{12x+21}{15}+\frac{3x-4}{15}=\frac{15x+17}{15}$.

2. $\frac{36a-48b}{84}-\frac{56a-28b+28c}{84}+\frac{91a-28c}{84}=\frac{71a-20b-56c}{84}$.

3. $\frac{24x-18y}{42}+\frac{9x+21y}{42}-\frac{10x-4y}{42}+\frac{9x+2y}{42}=\frac{32x+9y}{42}$.

4. $\frac{30x-20y}{50x}+\frac{25x-35y}{50x}+\frac{16x^2+4xy}{50x}=\frac{16x^2+55x+4xy-55y}{50x}$.

5. $\frac{16x^2-28y^2}{12x^2}+\frac{6x^2-16xy}{12x^2}+\frac{5x^2-2x^2y}{12x^2}=\frac{27x^2-2x^2y-16xy-28y^2}{12x^2}$.

6. $$\frac{180a^2+225b^2}{90b^2}+\frac{54ab+36b^2}{90b^2}+\frac{70b^2-20ab^2}{90b^2}=\frac{180a^2+54ab+331b^2-20ab^2}{90b^2}.$$

7. $$\frac{80x^3+100x^2}{60x^2}-\frac{36x^2-84x}{60x^2}+\frac{45}{60x^2}=\frac{80x^3+64x^2+84x+45}{60x^2}.$$

8. $$\frac{70a^2+28ab}{42ac}-\frac{84c^2-63bc}{42ac}+\frac{18ab-21bc}{42ac}=\frac{70a^2+46ab+42bc-84c^2}{42ac}, \text{ etc.}$$

9. $$\frac{2ac+5c^2}{a^2c^2}+\frac{4a^2c-3ac^2}{a^2c^2}-\frac{5ac-2c^2}{a^2c^2}=\frac{4a^2c-3ac^2-3ac+7c^2}{a^2c^2}, \text{ etc.}$$

10. $$\frac{3x^2y^2-4xy}{x^3y^3}-\frac{5x^2y^2+7x^2}{x^3y^3}-\frac{6x^2y^2-11y^2}{x^3y^3}=\frac{11y^2-8x^2y^2-4xy-7x^2}{x^3y^3}.$$

11. $$\frac{abc^2-b^2c^2}{a^3b^2c^2}+\frac{4a^2bc-5ab^2c}{a^3b^2c^2}+\frac{3a^4-7a^3b}{a^3b^2c^2}=\frac{3a^4-7a^3b+4a^2bc-5ab^2c+abc^2-b^2c^2}{a^3b^2c^2}.$$

L.

1. $$\frac{x+5+x-6}{(x-6)(x+5)}=\frac{2x-1}{(x-6)(x+5)}.$$

2. $$\frac{x-3-(x-7)}{(x-7)(x-3)}=\frac{4}{(x-7)(x-3)}.$$

3. $$\frac{1-x+1+x}{(1+x)(1-x)}=\frac{2}{(1+x)(1-x)}.$$

4. $$\frac{x^2+2xy+y^2-(x^2-2xy+y^2)}{(x+y)(x-y)}=\frac{4xy}{(x+y)(x-y)}.$$

5. $$\frac{1+x-2}{(1+x)(1-x)}=\frac{x-1}{(1+x)(1-x)}=\frac{-(1-x)}{(1+x)(1-x)}=\frac{-1}{1+x}$$

6. $$\frac{a(c+dx)-(ad-bc)x}{c(c+dx)}=\frac{ac+bcx}{c(c+dx)}=\frac{a+bx}{c+dx}.$$

7. $$\frac{x^2-xy+x^2+xy}{(x+y)(x-y)}=\frac{2x^2}{(x+y)(x-y)}.$$

8. $\dfrac{x-y+x}{(x-y)^2}=\dfrac{2x-y}{(x-y)^2}$.

9. $\dfrac{2(x+a)+3a}{(x+a)^2}=\dfrac{2x+5a}{(x+a)^2}$

10. $\dfrac{a-x+a+x}{2a(a+x)(a-x)}=\dfrac{2a}{2a(a+x)(a-x)}=\dfrac{1}{(a+x)(a-x)}$.

LI.

1. $\dfrac{1-a+1+a}{1-a^2}+\dfrac{2a}{1-a^2}=\dfrac{2+2a}{1-a^2}=\dfrac{2(1+a)}{1-a^2}=\dfrac{2}{1-a}$.

2. $\dfrac{1+x-(1-x)}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x+2x^3+2x-2x^3}{1-x^4}=\dfrac{4x}{1-x^4}$.

3. $\dfrac{x+x^2-x^2}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x+x^3+x-x^3}{1-x^4}=\dfrac{2x}{1-x^4}$.

4. $\dfrac{a+b-(a-b)}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2b}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2a^2b+2b^3-(2a^2b-2b^3)}{a^4-b^4}$

$\dfrac{4b^3}{a^4-b^4}-\dfrac{4b^3}{a^4+b^4}=\dfrac{4a^4b^3+4b^7-(4a^4b^3-4b^7)}{a^8-b^8}=\dfrac{8b^7}{a^8-b^8}$.

5. $\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x^2}{x(x+y)}=\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x}{x+y}=\dfrac{x^2+xy+y^2+xy}{y(x+y)}$

$=\dfrac{(x+y)^2}{y(x+y)}=\dfrac{x+y}{y}$.

6. $\dfrac{x^2-9+x^2-16}{(x+4)(x-3)}+\dfrac{x+5}{x+7}=\dfrac{2x^2-25}{(x+4)(x-3)}+\dfrac{x+5}{x+7}$

$=\dfrac{2x^3-25x+14x^2-175+(x+5)(x^2+x-12)}{(x+4)(x-3)(x+7)}$

$=\dfrac{2x^3-25x+14x^2-175+x^3+6x^2-7x-60}{(x+4)(x-3)(x+7)}$, etc.

7. $$\frac{x^2-4x+3+x^2-4x+4}{(x-2)(x-3)}+\frac{x-3}{x-4}=\frac{2x^2-8x+7}{(x-2)(x-3)}+\frac{x-3}{x-4}$$

$$=\frac{2x^3-16x^2+39x-28+(x-3)(x^2-5x+6)}{(x-2)(x-3)(x-4)}$$

$$=\frac{2x^3-16x^2+39x-28+x^3-8x^2+21x-18}{(x-2)(x-3)(x-4)}\text{, etc.}$$

8. $$\frac{3x-3a+4a}{(x-a)^2}-\frac{5a^2}{(x-a)^3}=\frac{(3x+a)(x-a)-5a^2}{(x-a)^3}=\frac{3x^2-2ax-a^2-5a^2}{(x-a)^3}$$

$$=\frac{3x^2-2ax-6a^2}{(x-a)^3}.$$

9. $$\frac{x+2-(x-1)}{(x-1)(x+2)}-\frac{3}{(x+1)(x+2)}=\frac{3}{(x-1)(x+2)}-\frac{3}{(x+1)(x+2)}$$

$$=\frac{3x+3-(3x-3)}{(x-1)(x+1)(x+2)}=\frac{6}{(x-1)(x+1)(x+2)}.$$

10. $$\frac{x+3}{(x+1)(x+2)(x+3)}-\frac{3}{(x+1)(x+2)(x+3)}=\frac{x}{(x+1)(x+2)(x+3)}.$$

11. $$\frac{x^2}{x^2-1}+\frac{x^2+x}{x^2-1}+\frac{x^2-x}{x^2-1}=\frac{3x^2}{x^2-1}.$$

12. $$\frac{a+e}{(a+c)(a+d)(a+e)}-\frac{a+d}{(a+c)(a+d)(a+e)}=\frac{e-d}{(a+c)(a+d)(a+e)}.$$

13. $$\frac{(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)}{(b+c)(c+a)(a+b)}$$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b+c)(c+a)(a+b)}=0.$$

14. $$\frac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}=\frac{x^2-2ax+a^2+x^2-2bx+b^2-a^2+2ab-b^2}{(x-a)(x-b)}$$

$$=\frac{2(x^2-ax-bx+ab)}{x^2-ax-bx+ab}=2.$$

15. $$\frac{(x+y)^2-2xy}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}=\frac{x^2+y^2}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}$$
$$=\frac{(x^2+y^2)(x-y)+x^2y-x^3}{y(x^2-y^2)}=\frac{x^3+xy^2-x^2y-y^3+x^2y-x^3}{y(x^2-y^2)}$$
$$=\frac{y^2(x-y)}{y(x^2-y^2)}=\frac{y}{x+y}.$$

16. $$\frac{(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)}{(b-c)(c-a)(a-b)}$$
$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b-c)(c-a)(a-b)}=0.$$

17. $$\frac{x(x-y)}{x^3-y^3}+\frac{2xy}{x^3-y^3}=\frac{x^2+xy}{x^3-y^3}.$$

18. $$\frac{2b-2c+2a-2b}{(a-b)(b-c)}+\frac{2}{c-a}=\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)};$$
$$\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$$
$$=\frac{(2ac-2a^2-2c^2+2ac+2ab-2b^2-2ac+2bc)+(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2)}{(a-b)(b-c)(c-a)}$$
$$=0.$$

19. $$\frac{(a+b)^2-2ab}{b(a+b)}+\frac{a^2b-a^3}{b(a^2-b^2)}=\frac{(a^2+b^2)(a-b)+a^2b-a^3}{b(a^2-b^2)}$$
$$=\frac{a^3+ab^2-a^2b-b^3+a^2b-a^3}{b(a^2-b^2)}=\frac{b^2(a-b)}{b(a^2-b^2)}=\frac{b}{a+b}.$$

20. $$\frac{(n+3)-1-(n+2)}{(n+1)(n+2)(n+3)}=0.$$

21. $$\frac{(a^2-bc)(b+c)+(b^2-ac)(a+c)+(c^2-ab)(a+b)}{(a+b)(a+c)(b+c)}$$
$$=\frac{a^2b+a^2c-b^2c-bc^2+ab^2-a^2c+b^2c-ac^2+ac^2-a^2b+bc^2-ab^2}{(a+b)(a+c)(b+c)}$$
$$=0.$$

LII.

1. $\dfrac{x}{x-y}+\dfrac{x-y}{y-x}=\dfrac{x}{x-y}+\dfrac{-x+y}{x-y}=\dfrac{y}{x-y}.$

2. $\dfrac{(3+2x)(2+x)}{4-x^2}-\dfrac{(2-3x)(2-x)}{4-x^2}+\dfrac{x^2-16x}{4-x^2}$

$$=\frac{6+7x+2x^2-(4-8x+3x^2)+x^2-16x}{4-x^2}=\frac{2-x}{4-x^2}=\frac{1}{2+x}.$$

3. $\dfrac{x}{x+1}-\dfrac{-x}{x-1}+\dfrac{x^2}{x^2-1}=\dfrac{x^2-x-(-x^2-x)+x^2}{x^2-1}=\dfrac{3x^2}{x^2-1}.$

4. $\dfrac{1}{6(y+1)}-\dfrac{1}{2(y-1)}+\dfrac{4}{3(1-y^2)}=\dfrac{y-1-3(y+1)}{6(y^2-1)}+\dfrac{-4}{3(y^2-1)}$

$$=\frac{y-1-3y-3-8}{6(y^2-1)}=\frac{-2y-12}{6(y^2-1)}=\frac{2y+12}{6(1-y^2)}=\frac{y+6}{3(1-y^2)}.$$

5. $\dfrac{1}{(m-2)(m-3)}+\dfrac{-2}{(m-1)(m-3)}+\dfrac{1}{(m-1)(m-2)}$

$$=\frac{m-1-2(m-2)+m-3}{(m-1)(m-2)(m-3)}=0.$$

6. $\dfrac{1}{(a-b)(x+b)}+\dfrac{-1}{(a-b)(x+a)}=\dfrac{x+a-(x+b)}{(a-b)(x+a)(x+b)}$

$$=\frac{a-b}{(a-b)(x+a)(x+b)}=\frac{1}{(x+a)(x+b)}.$$

7. $\dfrac{a^2+b^2}{(a+b)(a-b)}-\dfrac{2ab^2}{(a-b)(a^2+ab+b^2)}+\dfrac{2a^2b}{(a+b)(a^2-ab+b^2)}$

$$=\frac{(a^2+b^2)(a^2+ab+b^2)-2ab^2(a+b)}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$

$$=\frac{a^4+a^2b^2+a^3b+ab^3+a^2b^2+b^4-2a^2b^2-2ab^3}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$

$$=\frac{a^4+a^3b-ab^3+b^4}{(a+b)(a^3-b^3)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$

$$=\frac{(a^4+a^3b-ab^3+b^4)(a^2-ab+b^2)+2a^2b(a^3-b^3)}{(a+b)(a^3-b^3)(a^2-ab+b^2)}$$

$$=\frac{a^6+2a^2b^4-2ab^5+b^6+2a^5b-2a^2b^4}{(a^3-b^3)(a^3+b^3)}=\frac{a^6+2a^5b-2ab^5+b^6}{a^6-b^6}.$$

8. $$\frac{1}{4(1+x)}-\frac{-1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1-x-(-1-x)}{4(1-x^2)}+\frac{1}{2(1+x^2)}$$

$$=\frac{2}{4(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1}{2(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1+x^2+1-x^2}{2(1-x^4)}$$

$$=\frac{2}{2(1-x^4)}=\frac{1}{1-x^4}.$$

9. $$\frac{1}{(x-y)(y-z)}+\frac{-1}{(x-y)(x-z)}+\frac{1}{(x-z)(y-z)}$$

$$=\frac{(x-z)+(-y+z)+(x-y)}{(x-y)(y-z)(x-z)}=\frac{2(x-y)}{(x-y)(y-z)(x-z)}=\frac{2}{(x-z)(y-z)}.$$

10. $$\frac{1}{a(a-b)(a-c)}+\frac{-1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}$$

$$=\frac{bc(b-c)+ac(-a+c)+ab(a-b)}{abc(a-b)(a-c)(b-c)}$$

$$=\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}=\frac{1}{abc}.$$

LIII.

1. $$\frac{1}{(x+4)(x+5)}+\frac{1}{(x+7)(x+5)}=\frac{x+7+x+4}{(x+4)(x+5)(x+7)}$$

$$=\frac{2x+11}{(x+4)(x+5)(x+7)}.$$

2. $$\frac{1}{(x-7)(x-6)}+\frac{1}{(x-9)(x-6)}=\frac{x-9+x-7}{(x-7)(x-6)(x-9)}$$

$$=\frac{2(x-8)}{(x-6)(x-7)(x-9)}.$$

3. $$\frac{1}{(x+11)(x-4)}+\frac{1}{(x+11)(x-13)}=\frac{x-13+x-4}{(x-4)(x+11)(x-13)}$$

$$=\frac{2x-17}{(x-4)(x+11)(x-13)}.$$

4. $$\frac{1}{(x+1)(x+2)}+\frac{2x}{(x+1)(x+3)}+\frac{1}{(x+2)(x+3)}$$

$$=\frac{x+3+2x(x+2)+x+1}{(x+1)(x+2)(x+3)}=\frac{2x^2+6x+4}{(x+1)(x+2)(x+3)}$$

$$=\frac{2(x^2+3x+2)}{(x^2+3x+2)(x+3)}=\frac{2}{x+3}.$$

5. $$\frac{m^2+mn+2mn}{n(m+n)}-\frac{2mn}{(m+n)^2}=\frac{(m^2+3mn)(m+n)-2mn^2}{n(m+n)^2}$$

$$=\frac{m^3+3m^2n+m^2n+3mn^2-2mn^2}{n(m+n)^2}=\frac{m^3+4m^2n+mn^2}{n(m+n)^2}.$$

6. $$\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}-\frac{2}{1+x^2+x^4}$$

$$=\frac{1+x^3+1-x^3}{1+x^2+x^4}-\frac{2}{1+x^2+x^4}=0.$$

7. $$\frac{5+5x-6+6x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}-\frac{-7x}{3(1-x^2)}=\frac{11x-1+7x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}$$

$$=\frac{(18x-1)(1+x^2)+7x(1-x^2)}{3(1-x^4)}=\frac{18x+18x^3-1-x^2+7x-7x^3}{3(1-x^4)}$$

$$=\frac{11x^3-x^2+25x-1}{3(1-x^4)}.$$

8. $$\frac{(3-x)+2(x-1)}{8(x-1)(3-x)}+\frac{1}{8(x-5)}+\frac{1}{(1-x)(x-3)(x-5)}$$

$$=\frac{(x+1)(x-5)+(x-1)(3-x)}{8(x-1)(3-x)(x-5)}+\frac{1}{(x-1)(3-x)(x-5)}$$

$$=\frac{x^2-4x-5+4x-x^2-3+8}{8(x-1)(3-x)(x-5)}=0.$$

9. $\frac{(1-x+x^2-x^3)(1+x)+x^4}{1+x}=\frac{1-x^4+x^4}{1+x}=\frac{1}{1+x}.$

LIV.

1. Multiply by 2 ; $x=16$.

2. Multiply by 4 ; $3x=36$, $x=12$.

3. Multiply by 15 ; $5x+3x=120$; $8x=120$, $x=15$.

4. Multiply by 28 ; $7x-4x=84$; $3x=84$, $x=28$.

5. $-\frac{4x}{9}=-28$; $\frac{4x}{9}=28$, $\frac{x}{9}=7$, $x=63$.

6. Multiply by 15 ; $10x=528-12x$; $22x=528$; $x=24$.

7. Multiply by 12 ; $8x+48=7x+108$; $x=60$.

8. $\frac{2x}{3}+6=\frac{4x}{5}$; $10x+90=12x$; $x=45$.

9. $\frac{3x}{4}+3=\frac{5x}{6}$; $9x+36=10x$; $x=36$.

10. $\frac{7x}{8}=\frac{9x}{10}-3$; $35x=36x-120$; $x=120$.

11. $\frac{5x}{9}+\frac{7x}{12}=82$; $20x+21x=82\times 36$; $x=72$.

12. $\frac{x}{6}+\frac{x}{8}=28$; $4x+3x=28\times 24$; $x=96$.

13. $\frac{5x}{8}-\frac{3x}{4}=-8$; $5x-6x=-64$; $x=64$.

14. $9x+360-10x=348$; $-x=-12$; $x=12$.

15. $3x-44=2x-16$; $x=28$.

16. $6x+4x+3x=13$; $13x=13$; $x=1$.

17. Multiply by 70; $14x+28+10x-10=35x-70$; $-11x=-88$; $x=8$.

18. Multiply by 12; $6x+4x=117-3x$; $13x=117$; $x=9$.

19. Multiply by 140; $35x+315+40x=84x-168+420$; $-9x=-63$; $x=7$.

20. Multiply by 105; $357-63x=1015-385x+140x+70$; $182x=728$; $x=4$.

21. Multiply by 7; $2x-10=0$; $2x=10$; $x=5$.

22. Multiply by 329; $141x+188+28x-357=0$; $169x=169$; $x=1$.

23. Multiply by x; $3-3x=1-x$; $-2x=-2$; $x=1$.

24. Multiply by x; $12+x-5x=6$; $-4x=-6$; $x=\frac{3}{2}$.

25. Multiply by 20; $5x+2x+x=800$; $8x=800$; $x=100$.

26. Multiply by 8; $18x+12-4x=29x-348$; $-15x=-360$; $x=24$.

27. Multiply by $100x$; $275x-300=100-325x$; $600x=400$; $x=\frac{2}{3}$.

28. Multiply by 90; $225+540-30x=100x+30+27-18x+36$; $-112x=-672$; $x=6$.

29. Multiply by 12; $4x+3x-10x-144=20x-696$; $-23x=-552$; $x=24$.

30. Multiply by 60; $42x+12-720-45x=36x+156-255x$; $216x=864$; $x=4$.

LV.

1. Multiply by 2; $10x - x - 2 = 142$; $9x = 144$; $x = 16$.
2. Multiply by 3; $3x - 3 + x = 17$; $4x = 20$; $x = 5$.
3. Multiply by 4; $5 - 2x + 8 = 4x - 12x + 16$; $6x = 3$; $x = \frac{1}{2}$.
4. Multiply by 4; $10x - 5x = 9 - 6 + 2x$; $3x = 3$; $x = 1$.
5. Multiply by 30; $60x - 25x + 20 = 210 - 6 + 12x$; $23x = 184$; $x = 8$.
6. Multiply by 36; $18x + 36 = 56 - 27 - 45x$; $63x = -7$; $x = -\frac{1}{9}$.
7. Multiply by 48; $30x + 18 - 48 + 64x + 24x = 744 - 72 + 40x$; $78x = 702$; $x = 9$.
8. Multiply by 385; $55x + 275 - 77x + 154 = 35x + 315$; $-57x = -114$; $x = 2$.
9. Multiply by 105; $35x + 35 - 15x + 60 = 21x + 84$; $-x = -11$; $x = 11$.
10. Multiply by 24; $24x - 72 - 3x - 6 = 8x$; $13x = 78$; $x = 6$.
11. Multiply by 84; $12x + 60 = 21x + 42 - 28x + 56$; $19x = 38$; $x = 2$.
12. Multiply by 33; $11x - 3x + 3 = 33x - 297$; $-25x = -300$; $x = 12$.
13. Multiply by 70; $14x + 28 = 35x - 70 - 10x + 10$; $-11x = -88$; $x = 8$.
14. Multiply by 140; $35x + 315 - 84x + 168 = 420 - 40x$; $-9x = -63$; $x = 7$.
15. Multiply by 78; $39x + 39 - 26x + 78 = 6x + 180$; $7x = 63$; $x = 9$.
16. Multiply by 35; $10x - 7x - 21 = 105x - 735$; $-102x = -714$; $x = 7$.

17. Multiply by 154 ; $44x+154-126x+112=77x-847$;
$-159x=-1113$; $x=7$.

18. Multiply by 572 ; $1001x-4433-176-330x=182x-208$;
$489x=4401$; $x=9$.

19. Multiply by 273 ; $728x-1365-429x+39=147x+42$;
$152x=1368$; $x=9$.

20. Multiply by 56 ; $49x+63-24x-8=126x-182-996+36x$;
$-137x=-1233$; $x=9$.

21. Multiply by 280 ; $28x+2800x=140x+56x+7x-400+40x+$
26250 ; $2585x=25850$; $x=10$.

LVI.

1. $(a+b)x=c$; $x=\dfrac{c}{a+b}$.

2. $5bx-cx=3c-2a$; $(5b-c)\,x=3c-2a$, etc.

3. $ax+fx=a^2b-bc+d$; $(a+f)\,x=a^2b-bc+d$, etc.

4. $ax-5x=bc-dm$; $(a-5)x=bc-dm$, etc.

5. $-a^2x-ax=-a^2b-abc$; $ax+x=ab+bc$; $(a+1)\,x=b\,(a+c)$, etc.

6. $3acx-12cdx=abc+6bcd$; $(3a-12d)\,x=ab+6bd$, etc.

7. $3ackx-kx+ackx=-k^2-3k-k^2+3abk$;
$3acx-x+acx=-k-3-k+3ab$; $(4ac-1)x=3ab-2k-3$;
$x=\dfrac{3ab-2k-3}{4ac-1}$.

8. $abcx-cmx+ac^2x=ac^2+abc-mc$;
$abx-mx+acx=ac+ab-m$; $x=1$.

9. $(a+b)^2-x^2=ab+(b-a)x-x^2-ab$; $x=\frac{(a+b)^2}{b-a}$.

10. $a^2-x^2=2a^2+2ax-x^2$; $-2ax=a^2$; $x=-\frac{a}{2}$.

11. $a^4+2a^2x+x^2=x^2+4a^2+a^4$; $2a^2x=4a^2$; $x=2$.

12. $a^4-x^2=a^4+2ax-x^2$; $-2ax=0$; $x=0$.

13. $ax-b+ac=x+ac$; $ax-x=b$; $x=\frac{b}{a-1}$.

14. $2ax-3a+bx=1$; $2ax+bx=3a+1$; $x=\frac{3a+1}{2a+b}$.

15. $18a-4ax+2b=3x$; $3x+4ax=18a+2b$; $x=\frac{18a+2b}{4a+3}$.

16. $ax^2-bx-1=ax^2-a$; $-bx=1-a$; $x=\frac{a-1}{b}$.

17. $mp^2x+mx^3=mpqx^2+mx^3$; $mp^2x=mpqx^2$; $p=qx$; $x=\frac{p}{q}$.

18. $dx-abd=ac-adx$; $dx+adx=abd+ac$; $x=\frac{abd+ac}{ad+d}$.

19. $x^2-a-ax+x^2=2x^2-ab$; $-ax=a-ab$; $x=b-1$

20. $3bx-abc+cx^2=4bx-abc$; $cx^2=bx$; $x=\frac{b}{c}$.

21. $a^3b+a^2x-b^3+bx=b^2x-b^3-a^3b+a^2x$; $bx-b^2x=-2a^3b$;

$x=\frac{2a^3}{b-1}$.

22. $6ax-4b-3ax+3a=6ax-4b$; $-3ax=-3a$; $x=1$.

23. $abm^2-b^2m-amx+bx=0$; $(b-am)x=bm(b-am)$; $x=bm$.

24. $2a^3b^4-b^3x+3a^3bc=3a^2cx(a+b)-ab^4+2a^2b^3x$;

$-b^3x-3a^3cx-3a^2bcx-2a^2b^3x=-3a^3bc-2a^3b^4-ab^4$;

$x=\frac{3a^3bc+2a^3b^4+ab^4}{b^3+3a^3c+3a^2bc+2a^2b^3}$.

25. $acx^2 + abc - ac^2x + abx - acx^2 = 0$; $ac^2x - abx = abc$; $x = \frac{bc}{c^2 - b}$.

26. $ad^2 + ax^2 = acdx + ax^2$; $x = \frac{d}{c}$.

27. $ab = (bc + d)\, x + 1$; $x = \frac{ab - 1}{bc + d}$.

28. $3ac + cx = 3a^2 + ax + am - mx$; $cx - ax + mx = am - 3ac + 3a^2$;

$$x = \frac{am - 3ac + 3a^2}{c - a + m}.$$

29. $ab + ax + bx + x^2 - ab - ac = \frac{a^2c}{b} + x^2$; $ax + bx = ac + \frac{a^2c}{b}$;

$$(a + b)x = \frac{(ab + a^2)\, c}{b}\ ;\ x = \frac{ac}{b}.$$

30. $a^2ce - a^2dx - 2abdx - b^2dx - abdx = a^2de - 3abdx$;

$$-a^2dx - b^2dx = a^2de - a^2ce\ ;\ \ x = \frac{a^2e\,(c - d)}{(a^2 + b^2)\, d}.$$

LVII.

1. $(3x + 7)(4x + 3) = (3x + 5)(4x + 5)$;
$12x^2 + 37x + 21 = 12x^2 + 35x + 25$; $2x = 4$; $x = 2$.

2. $(x + 6)(2x - 5) = x(2x + 5)$; $2x^2 + 7x - 30 = 2x^2 + 5x$; $2x = 30$; $x = 15$.

3. $(2x + 7)(2x - 1) = (4x - 1)(x + 2)$; $4x^2 + 12x - 7 = 4x^2 + 7x - 2$;
$5x = 5$; $x = 1$.

4. $(5x - 1)(2x - 3) = (5x - 3)(2x + 3)$; $10x^2 - 17x + 3 = 10x^2 + 9x - 9$;

$$-26x = -12\ ;\ x = \frac{6}{13}.$$

5. $4x - 3 + 2(3x - 2) = 0$; $4x - 3 + 6x - 4 = 0$; $10x = 7$; $x = \frac{7}{10}$.

6. $2(1-2x)-5(1-5x)=0$; $2-4x-5+25x=0$; $21x=3$; $x=\frac{1}{7}$.

7. $\frac{x+1+x-1}{x^2-1}=\frac{3}{x^2-1}$; $2x=3$; $x=\frac{3}{2}$.

8. $\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{4x+3}{9}$; $\frac{7x-29}{5x-12}=\frac{8x+19-8x-6}{18}$;

$18(7x-29)=13(5x-12)$; $126x-522=65x-156$; $61x=366$;

$x=6$.

9. $\frac{x}{3}-\frac{2}{3}=\frac{x^2-5x}{3x-7}$; $(x-2)(3x-7)=3(x^2-5x)$;

$3x^2-13x+14=3x^2-15x$; $2x=-14$; $x=-7$.

10. $\frac{(3x+2)(x+2)+(2x-4)(x-1)}{x^2+x-2}=5$;

$3x^2+8x+4+2x^2-6x+4=5x^2+5x-10$; $-3x=-18$; $x=6$.

11. Multiply by 210; $35(x+3)-30(11-x)=84(x-4)-10(x-3)$; $35x+105-330+30x=84x-336-10x+30$; $-9x=-81$; $x=9$.

12. $(x+1)(2x+2)=2(x-3)(x+6)$; $2x^2+4x+2=2x^2+6x-36$; $-2x=-38$; $x=19$.

13. $\frac{3x^2(2x+3)+2x+1}{6x^2+3x}=x+1$; $6x^3+9x^2+2x+1=(6x^2+3x)(x+1)$;

$6x^3+9x^2+2x+1=6x^3+9x^2+3x$; $-x=-1$; $x=1$.

14. $\frac{3(x-1)-(x+1)(x+1)}{x^2-1}=\frac{-x^2}{x^2-1}$; $3x-3-x^2-2x-1=-x^2$; $x=4$.

15. $\frac{2(1+x)+8(1-x)}{1-x^2}=\frac{45}{1-x^2}$; $2+2x+8-8x=45$; $-6x=35$;

$x=-\frac{35}{6}$.

16. $\frac{4}{x-8}+\frac{3}{2(x-8)}-\frac{29}{24}=\frac{2}{3(x-8)}$; multiply by $24(x-8)$;

$96+36-29x+232=16$; $-29x=-348$; $x=12$.

17. $x^4-4x^2+20x-24=x^4-4x^2+16x-16$; $4x=8$; $x=2$.

18. $2x^4+2x^3-9x^2+12=2x^4+2x^3-23x^2+7x+12$; $14x^2=7x$; $x=\frac{1}{2}$.

19. Multiply by $16x$; $4x^2-16x=4x^2-3x-\frac{13}{8}$; $-13x=-\frac{13}{8}$; $x=\frac{1}{8}$

20. $5-x\left(\frac{7x-4}{2x}\right)=\frac{x}{2}-\frac{8x-4}{4}$; $5-\frac{7x-4}{2}=\frac{x}{2}-\frac{4x-2}{2}$;

$10-7x+4=x-4x+2$; $-4x=-12$; $x=3$.

LVIII.

1. $\frac{5x}{10}-2=\frac{25x}{100}+\frac{2x}{10}-1$; $50x-200=25x+20x-100$; $x=20$.

2. $\frac{325x}{100}-\frac{51}{10}+x-\frac{75x}{100}=\frac{39}{10}+\frac{5x}{10}$; $325x-510+100x-75x=390+50x$

$300x=900$; $x=3$.

3. $\frac{125x}{1000}+\frac{x}{100}=13-\frac{2x}{10}+\frac{4}{10}$; $125x+10x=13000-200x+400$;

$335x=13400$; $x=40$.

4. $\frac{3x}{10}+\frac{1305x}{1000}+\frac{5x}{10}=\frac{2295}{100}-\frac{195x}{1000}$;

$300x+1305x+500x=22950-195x$; $2300x=22950$; $x=\frac{459}{46}$.

5. $\frac{2x}{10}-\frac{x}{100}+\frac{5x}{1000}=\frac{117}{10}$; $200x-10x+5x=11700$;

$195x=11700$; $x=60$.

6. $\frac{24x}{10}-\frac{36x-5}{50}=\frac{8x}{10}+\frac{89}{10}$; $120x-36x+5=40x+445$; $44x=440$; $x=10$.

7. $\frac{24x}{10}-\frac{1075}{100}=\frac{25x}{100}$; $240x-1075=25x$; $215x=1075$; $x=5$.

8. $\frac{5x}{10}+2-\frac{75x}{100}=\frac{4x}{10}-11$; $50x+200-75x=40x-1100$; $-65x=-1300$; $x=20$.

9. $\frac{405}{900x}=\cdot150$; $\frac{45}{100x}=\cdot15$; $45=15x$; $x=3$.

10. $\frac{25x}{10}-\frac{2+x}{7}\times\frac{-7}{4}=\frac{5}{10}-\frac{5x+3}{8}$; $\frac{5x}{2}+\frac{2+x}{4}=\frac{1}{2}-\frac{5x+3}{8}$; $20x+4+2x=4-5x-3$; $27x=-3$; $x=-\frac{1}{9}$

11. $\frac{85}{20}-\frac{2}{10x}=\frac{17}{4}-\frac{10-x}{10x}$; $85x-4=85x-20+2x$; $-2x=-16$; $x=8$.

12. $\frac{48x}{600}-\frac{30-40x}{2}=1993$; $8x-1500+2000x=199300$; $2008x=200800$; $x=100$.

13. $\frac{20-30x}{15}+\frac{500x}{125}-\frac{2x-3}{9}=\frac{10x-20}{18}+\frac{25}{9}$;
$\frac{4-6x}{3}+4x-\frac{2x-3}{9}=\frac{5x-10}{9}+\frac{25}{9}$;
$12-18x+36x-2x+3=5x-10+25$; $11x=0$; $x=0$.

14. $\frac{2408}{100x}+\frac{1}{x}\times\frac{4}{100}\times\frac{10x+9}{10}=\frac{2412}{10}$; $\frac{2408}{100x}+\frac{40x+36}{1000x}=\frac{2412}{10}$;
$24080+40x+36=241200x$; $24116=241160x$; $x=\cdot1$.

15. $\frac{5x}{10}+\frac{45x-75}{60}=\frac{12}{2}-\frac{3x-6}{9}$; $\frac{x}{2}+\frac{3x-5}{4}=6-\frac{x-2}{3}$;
$6x+9x-15=72-4x+8$; $19x=95$: $x=5$.

16. $\frac{1}{2}-\frac{35x}{10x-20}-\frac{24-3x}{8}=\frac{375x}{1000}$; $500-\frac{3500x}{x-2}-3000+375x=375x$;

$$-2500=\frac{3500x}{x-2};\ -25x+50=35x;\ 50=60x;\ x=\frac{5}{6}.$$

17. $\frac{15x}{100}+\frac{135x-225}{600}=\frac{36}{20}-\frac{9x-18}{90}$; $\frac{3x}{20}+\frac{45x-75}{200}=\frac{18}{10}-\frac{x-2}{10}$;

$$30x+45x-75=360-20x+40;\ 95x=475;\ x=5.$$

LIX.

1. Let x be the number; then $\frac{x}{2}+\frac{x}{4}+\frac{x}{5}=95$, etc.

2. Let x be the number; then $\frac{x}{12}+\frac{x}{20}+\frac{x}{40}=38$, etc.

3. Let x be the number; then $\frac{x}{4}-\frac{x}{5}=4$, etc.

4. Let x be the number; then $\frac{x}{25}-\frac{x}{35}=8$, etc.

5. Let x be one part; then $60-x$ is the other; and $\frac{x}{7}=\frac{60-x}{8}$, etc.

6. Let x be one part; then $50-x$ is the other;

and $\frac{x}{4}+\frac{5(50-x)}{6}=40$, etc.

7. Let x be the greater; then $100-x$ is the less; and

$\frac{x}{4}-\frac{100-x}{3}=11$, etc.

8. Let x be the number; then $x-\left(\frac{x}{3}+\frac{x}{10}+\frac{x}{12}\right)=58$, etc.

9. Let x be the number; then $33-\frac{x}{4}-\frac{x}{5}-\frac{x}{10}=0$, etc.

10. Let x be the number; then $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=\frac{x}{2}+112$, etc.

11. Let x be the number; then $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{12}+30=2x$, etc.

12. Let x be the greater; then $x-8$ is the less; and $\frac{x}{x-8}=3$, etc.

13. Let x be the property in pounds; then $\frac{x}{7}=x-1626$, etc.

14. Let x be the greater; then $x-504$ is the less; and $\frac{x}{x-504}=15$, etc.

15. Let x be the greater; then $5760-x$ is the less; and

$$x-(5760-x)=\frac{x}{3}\text{ etc.}$$

16. Let x be the number; then $x+\frac{x}{2}-60=65-x$, etc.

17. Let x be the greater; then $x-20$ is the less; and $\frac{x}{7}=\frac{x-20}{3}$, etc.

18. Let x be the less; then $31207-x$ is the greater; and

$$\frac{31207-x}{x}=15+\frac{1335}{x}\text{, etc.}$$

19. Let x be the age of the younger; then $27-x$ is the age of the elder;

and $\frac{27-x}{x}=\frac{7}{2}$, etc.

20. Let x be the greater; then $237-x$ is the less; and $237-x=\frac{4x}{5}$, etc.

21. Let x be A's share; then $\frac{2x}{7}$ is B's share; and $x+\frac{2x}{7}=1800$, etc.

22. Let x be one part; then $46-x$ is the other; and $\frac{x}{7}+\frac{46-x}{3}=10$, etc.

23. Let x be one part; then $a-x$ is the other; and $\frac{x}{m}+\frac{a-x}{n}=b$, etc.

24. Let x be one number; then $a-x$ is the other; and $x-(a-x)=b$, etc.

25. Let x be the number; then $\frac{4x}{3}=24$, etc.

26. Let x be B's share; then $\frac{5x}{11}$ is A's share; and $x+\frac{5x}{11}$ is C's share; then $x+\frac{5x}{11}+x+\frac{5x}{11}=864$, etc.

27. Let x be the property; then $\frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600=x$, etc.

28. Let x be the second; then $70-x$ is the first; and $\frac{70-x}{x}=2+\frac{1}{x}$, etc.

29. Let x be the first; then $x+25$ is the second; and $\frac{x+25}{x}=4+\frac{4}{x}$.

30. Let x be the greater; then $208-x$ is the less; and $\frac{x}{4}+\frac{208-x}{3}=4\{x-(208-x)\}-4$, etc.

31. Let x be the number of days; then $\frac{2x}{3}-\frac{x}{2}=13$, etc.

32. Let x be the number of gallons; then $\frac{4x}{5}-10=\frac{2x}{3}$.

33. In 25 years the sum of their ages will be increased by 50 years; hence the sum of their ages at the present time is 50 years. Let x be the age of the father; then $50-x$ is the age of the

son; and the difference of their ages is $x-(50-x)$, or, $2x-50$. In 20 years the age of the father will be $x+20$, and the age of the son $70-x$. Hence the sum in 20 years will be $x+20+(70-x)$, or, 90 years. Therefore, by the question, $2x-50=30$, or, $x=40$.

34. Let x be the number of years; then $70-x=\frac{10}{3}(35-x)$, etc.

35. Let x be the number of years; then $72-x=5(48-x)$, etc.

36. Let x be the number of days; then $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$, etc.

37. Let x be the number of days; then $\frac{1}{50}+\frac{1}{60}+\frac{1}{75}=\frac{1}{x}$, etc.

38. Let x be the number of days; then $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ represents the part done daily by *two* men of A's strength, *two* of B's, and *two* of C's, working together. Hence A, B, and C together do $\frac{1}{2}$ of $\left(\frac{1}{12}+\frac{1}{15}+\frac{1}{20}\right)$ daily.

$$\therefore\ \frac{1}{x}=\frac{1}{2}\text{ of }\frac{5+4+3}{60},\text{ or, }x=\frac{120}{12}=10.$$

39. Let x be the number of hours; then $\frac{1}{x}$ is the part A does hourly;

A and B do $\frac{1}{4}$ hourly; A and C do $\frac{5}{18}$ hourly;

$\therefore$ B does $\frac{1}{4}-\frac{1}{x}$, and C does $\frac{5}{18}-\frac{1}{x}$ hourly.

$\therefore$ B and C do $\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$ hourly.

Hence $\frac{7}{36}=\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$, etc.

40. Let x be the number of days; then $\frac{2}{5}+\frac{3}{10}+\frac{4}{15}=\frac{1}{x}$, etc.

41. Let x be the number of days; then B will do the whole in $3x$ days.

$\therefore$ B does $\frac{1}{3x}$ daily. Now A does $\frac{3}{50}$ daily, and in 3 days he does $\frac{9}{50}$, while B in 3 days does $\frac{1}{x}$.

Hence $\frac{9}{50}+\frac{1}{x}=\frac{2}{5}$, etc.

42. Let x be the number of hours; then $\frac{1}{3}+\frac{1}{4}=\frac{1}{x}$, etc.

43. Let x be the number of hours; then $\frac{3}{4}+\frac{3}{10}+\frac{1}{5}=\frac{1}{x}$, etc.

44. Let x be the number of minutes; then $\frac{1}{40}-\frac{1}{60}=\frac{1}{x}$, etc.

45. Let x be the number of minutes; then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$, etc.

46. The first pipe lets in 12 gall. in $\frac{13}{4}$ min., or, $\frac{48}{13}$ gall. in a min.

The second „ $\frac{46}{3}$ gall. in $\frac{5}{2}$ min., or, $\frac{92}{15}$ gall. in a min.

The third „ 17 gall. in 3 min., or, $\frac{17}{3}$ gall. in a min.

$\therefore$ All three together let in $\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)$ gall. in a min.

Let x be the number of minutes in which they fill the vessel;

then $x\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)=755\frac{1}{4}$,

or, $\frac{2160x+3588x+3315x}{585}=\frac{3021}{4}$, or, $\frac{9063x}{585}=\frac{3021}{4}$,

or $\frac{3x}{585}=\frac{1}{4}$; $\therefore$ $x=\frac{585}{12}=48\frac{3}{4}$.

47. Let x be the number of gallons let in per minute by the third pipe; then $x+10$ and $x-4$ are the number of gallons let in by the other pipes; then $15(x+x+10+x-4)=2400$, etc.

48. Let x be the distance from Ely; then $16-x$ is the distance from Cambridge; and $\frac{x}{4\frac{4}{9}}=\frac{16-x}{\frac{60}{18}}$; $\frac{9x}{40}=\frac{9(16-x)}{30}$; $3x=64-4x$; $x=9\frac{1}{7}$.

49. Let x be the distance up; then x is the distance down; and $\frac{x}{2\frac{1}{3}}+\frac{x}{3\frac{1}{2}}=5$; $\frac{3x}{7}+\frac{2x}{7}=5$; $x=7$; and $\therefore 2x=14$.

50. In 1 hour he walks $\frac{a}{b}$ miles; hence in c hours he walks $\frac{ac}{b}$ miles.

He walks 1 mile in $\frac{b}{a}$ hours; hence he walks d miles in $\frac{bd}{a}$ hours.

51. Let x be the number of days; then $286x=244(x+2)$, etc.

52. Let x be the number of hours; then $x\times\frac{22\frac{1}{2}}{3}=(x+8)\times\frac{31\frac{1}{2}}{5}$; $\frac{15x}{2}=\frac{63(x+8)}{10}$, etc.

53. Let x be the distance from Cambridge; $60-x$ is the distance from London; then $\frac{x}{4}=\frac{60-x}{3\frac{3}{4}}$; $\frac{x}{4}=\frac{240-4x}{15}$, etc.

54. Let x be the number of hours; then $\frac{5}{3}\times x=\frac{7}{5}(x+8)$; $x=42$. Hence 50 hours after A started.

55. (1.) Let the time be x minutes past 1; then $x=5+\frac{x}{12}+30$, etc.

(2.) Let the time be x minutes past 4; then $x=20+\frac{x}{12}+30$, etc.

(3.) Let the time be x minutes past 8; then $x+30=40+\frac{x}{12}$, etc.

56. (1.) Let the time be x minutes past 2; then $x=10+\frac{x}{12}+15$, etc.

(2.) Let the time be x minutes past 4; then there are two solutions, (α) $x+15=20+\frac{x}{12}$, etc. (β) $x=20+\frac{x}{12}+15$, etc.

(3.) Let the time be x minutes past 7; then there are two solutions, (α) $x+15=35+\frac{x}{12}$, etc. (β) $x=35+\frac{x}{12}+15$, etc.

57. (1.) Let the time be x minutes past 3; then $x=15+\frac{x}{12}$, etc.

(2.) Let the time be x minutes past 6; then $x=30+\frac{x}{12}$, etc.

(3.) Let the time be x minutes past 9; then $x=45+\frac{x}{12}$, etc.

58. Let $2x$ be the number of apples; he pays for them $\frac{4x}{5}$ pence.

He sells x for $\frac{x}{2}$ pence, and x for $\frac{x}{3}$ pence.

Hence $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

59. Let x be the number of sovereigns he had at first.

Then he has $\frac{x}{2}-\frac{1}{2}$ left after the first gift;

and he has $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}$ left after the second gift.

Then $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}=0$; $\frac{x-1}{4}=\frac{1}{2}$; $x=3$.

60. $\dfrac{\frac{2}{3}+n}{3n+\frac{69}{3}}=\frac{1}{33}$; $\frac{2+3n}{9n+69}=\frac{1}{33}$; $66+99n=9n+69$, etc.

61. Let x be the number of days; then $x-3$ is the number of days that the pursuing army is in motion; and $18x+25=23(x-3)$, etc.

62. Let x be his original income in pounds;

then $x-\frac{x}{40}-\frac{1}{13}\left(x-\frac{x}{40}\right)=540$, etc.

63. Let x be the original sum; then first remainder is $\frac{x}{2}-50$, or, $\frac{x-100}{2}$;

second remainder is $\frac{4}{5}$ of $\frac{x-100}{2}-30$, or, $\frac{2x-200-150}{5}$;

third remainder is $\frac{3}{4}$ of $\frac{2x-350}{5}-20$, or, $\frac{6x-1050-400}{20}$;

thus $\frac{6x-1450}{20}=10$; $6x=1650$; $x=275$.

64. Let $2x$ be the number of eggs bought; for x I gave $\frac{x}{2}$ pence, and for x I gave $\frac{x}{3}$ pence; and I sold $2x$ eggs for $\frac{4x}{5}$ pence.

Thus $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

65. Let x be the number of minutes in which C would fill it; then $\frac{1200}{x}$ = number of gall. let in by C each minute; and therefore $\frac{1200}{x}+10$ = number of gall. let in by A and B each minute.

Hence $\frac{1200}{x}+10+\frac{1200}{x}=\frac{1200}{24}$; whence $x=60$.

Thus A takes 90 minutes and C 60 minutes.

Also B lets in $\left(\frac{1200}{24}-\frac{1200}{90}-\frac{1200}{60}\right)$ gall. per minute, or,

$50-\frac{40}{3}-20$, or, $\frac{50}{3}$ gall. per minute.

$\therefore$ B takes $1200\div\frac{50}{3}$ minutes, or, 72 minutes to fill it.

66. Let x be the number of days B would take; then $2x$ is the number A would take.

Then $\frac{1}{x}+\frac{1}{2x}$ is the part A and B drink daily,

$\therefore$ $\frac{3}{4}$ of $\left(\frac{1}{x}+\frac{1}{2x}\right)$ is the part C drinks daily.

Then $\frac{1}{x}+\frac{1}{2x}+\frac{3}{4x}+\frac{3}{8x}=\frac{1}{24}$; $x=63$, etc.

67. Let x be the number of shots each fires.

Then $\frac{7x}{12}$ is the number of A's hits, and $\frac{9x}{12}$ of B's hits.

Thus $\frac{7x}{12}+\frac{9x}{12}=32$; $x=24$.

68. Let x be the number of horses; then $2x$ is the number of oxen, and $100-3x$ is the number of sheep. Reducing all the prices to shillings,

$440x+250\times 2x+30(100-3x)=4700$; $x=2$, etc.

69. Let x be the whole number of marks.

Then $\frac{x}{5}$ = number of marks obtained by B.

And $\frac{2x}{5}$ = number of marks assigned to book work in the paper.

Also $\frac{2x}{5}$ = number of marks obtained by A for riders.

$\therefore$ $\frac{2x}{5}+\frac{2x}{5}=160$; $x=200$.

70. Since the man made a mistake of 55 minutes, the distance between the hands was 5 minutes.

Let x be the number of minutes past 2.

Then $x+5=10+\frac{x}{12}$; $x=5\frac{5}{11}$.

71. Let x be the original force; then $\frac{5x}{6}-4000$ = remainder after the defeat.

Hence $\frac{5x}{6}-1000=$ number after the reinforcement.

Then $\frac{3}{4}\left(\frac{5x}{6}-1000\right)=18000$; $\frac{5x}{8}-750=18000$; $x=30000$.

72. Let x be the original debt; then $x+\frac{x}{4}-25000000=$ debt after the peace.

Then $\frac{4\frac{1}{2}\times x}{100}=\frac{4\left(x+\frac{x}{4}-25000000\right)}{100}$; $\frac{9x}{2}=5x-100000000$, etc.

73. The influx goes on for 10 days and 15 hours, or, 255 hours.

The consumption extends over 10 days (excluding Sunday) of 15 hours each, or, 150 hours.

Hence if x be the number of gallons that come in each hour,

$$255x+2250=150\times 2x;\ x=50.$$

LX.

1. $a+x+\frac{3a}{x}=\frac{ax+x^2+3a}{x}$.

2. $\frac{a^2+ax}{x^2}-\frac{2x-2a}{x}=\frac{a^2+ax-2x^2+2ax}{x^2}=\frac{a^2+3ax-2x^2}{x^2}$.

3. $\frac{x-y}{x}+\frac{2y}{x-y}=\frac{x^2-2xy+y^2+2xy}{x(x-y)}=\frac{x^2+y^2}{x(x-y)}$.

4. $\frac{4a+4b}{a-b}-\frac{2a^2-2b^2}{a^2+b^2}=\frac{4a^3+4ab^2+4a^2b+4b^3-2a^3+2ab^2+2a^2b-2b^3}{(a-b)(a^2+b^2)}$, etc.

LXI.

1. $\frac{5x+4}{35}-\frac{23x}{70}=\frac{10x+8-23x}{70}=\frac{8-13x}{70}$.

2. $\frac{\frac{x^2-y^2}{xy}}{x-y}=\frac{x^2-y^2}{xy(x-y)}=\frac{x+y}{xy}$.

3. $\dfrac{1-x^2}{\dfrac{x+1}{x}}=\dfrac{x(1-x^2)}{1+x}=x(1-x).$

4. $\dfrac{y\left(\dfrac{x+y}{y}\right)}{x\left(\dfrac{x-y}{x}\right)}=\dfrac{x+y}{x-y}.$

5. $\dfrac{5x^2+x^3+1}{x^2}\div\dfrac{2x^2-x^3+1}{x^2}=\dfrac{5x^2+x^3+1}{2x^2-x^3+1}.$

6. $\dfrac{x^3+1}{x^2}\div\dfrac{x+1}{x}=\dfrac{(x+1)(x^2-x+1)x}{x^2(x+1)}=\dfrac{x^2-x+1}{x}.$

7. $\dfrac{a^3-1}{a^2}\div\dfrac{a-1}{a}=\dfrac{(a-1)(a^2+a+1)a}{a^2(a-1)}=\dfrac{a^2+a+1}{a}.$

8. $\dfrac{x^2-ax+x^2+ax}{x^2-a^2}\div\dfrac{2x}{x^2-a^2}=\dfrac{2x^2}{2x}=x.$

9. $\dfrac{2x}{x^2+x^2}=\dfrac{2x}{2x^2}=\dfrac{1}{x}.$

10. $\dfrac{x^2}{1+x}+1-\dfrac{1}{1+x}=\dfrac{x^2+1+x-1}{1+x}=\dfrac{x(1+x)}{1+x}=x.$

11. $\dfrac{x^2-2xy+y^2+x^2+2xy+y^2}{x^2-y^2}\div\dfrac{x^2-2xy+y^2-x^2-2xy-y^2}{x^2-y^2}$

$=\dfrac{2x^2+2y^2}{-4xy}$, etc.

12. $\dfrac{1+x+x^2}{1}\div\dfrac{x^2+x+1}{x^2}=x^2.$

13. $\dfrac{a^2+2ab+b^2+b^2}{b(a+b)}\div\dfrac{a+b}{ab}=\dfrac{(a^2+2ab+2b^2)ab}{b(a+b)(a+b)}$, etc.

14. $\dfrac{2m^2-3m+1}{m}\div\dfrac{2m-1}{m}=\dfrac{2m^2-3m+1}{2m-1}=m-1.$

15. $\dfrac{c+b+a}{abc}\div\dfrac{(a+b+c)(a-b-c)}{ab}=\dfrac{ab(a+b+c)}{abc(a+b+c)(a-b-c)}$, etc.

LXII.

1. $\frac{a^4}{2a^4}+\frac{3a^3}{2a^4}+\frac{2a^2}{2a^4}+\frac{5a}{2a^4}=\frac{1}{2}+\frac{3}{2a}+\frac{1}{a^2}+\frac{5}{2a^3}$.

2. $\frac{a^2bc}{abcd}+\frac{ab^2d}{abcd}+\frac{abc^2}{abcd}+\frac{bcd^2}{abcd}=\frac{a}{d}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$.

3. $\frac{x^3}{x^2y^2}-\frac{3x^2y}{x^2y^2}+\frac{3xy^2}{x^2y^2}-\frac{y^3}{x^2y^2}=\frac{x}{y^2}-\frac{3}{y}+\frac{3}{x}-\frac{y}{x^2}$.

4. $\frac{9a^3}{108}-\frac{12a^2}{108}+\frac{6a}{108}-\frac{3}{108}=\frac{a^3}{12}-\frac{a^2}{9}+\frac{a}{18}-\frac{1}{36}$.

5. $\frac{18p^2}{3pqrs}+\frac{12q^2}{3pqrs}-\frac{36r^2}{3pqrs}+\frac{72s^2}{3pqrs}=\frac{6p}{qrs}+\frac{4q}{prs}-\frac{12r}{pqs}+\frac{24s}{pqr}$.

6. $\frac{10x^3}{1000}-\frac{25x^2}{1000}+\frac{75x}{1000}-\frac{125}{1000}=\frac{x^3}{100}-\frac{x^2}{40}+\frac{3x}{40}-\frac{1}{8}$.

LXIII.

1.
$$
\begin{array}{l}
1+a)\,2\,(2-2a+2a^2-2a^3+2a^4\ldots \\
\quad\ \underline{2+2a} \\
\qquad -2a \\
\qquad \underline{-2a-2a^2} \\
\qquad\qquad\quad 2a^2 \\
\qquad\qquad\quad \underline{2a^2+2a^3} \\
\qquad\qquad\qquad -2a^3 \\
\qquad\qquad\qquad \underline{-2a^3-2a^4} \\
\qquad\qquad\qquad\qquad\quad 2a^4 \\
\qquad\qquad\qquad\qquad\quad \underline{2a^4+2a^5} \\
\qquad\qquad\qquad\qquad\qquad -2a^5
\end{array}
$$

2. $m+2)\,m\,(1-\frac{2}{m}+\frac{4}{m^2}-\frac{8}{m^3}+\frac{16}{m^4}\cdots$

$$\begin{array}{l}
\underline{m+2} \\
-2 \\
\underline{-2-\frac{4}{m}} \\
\frac{4}{m} \\
\underline{\frac{4}{m}+\frac{8}{m^2}} \\
-\frac{8}{m^2} \\
\underline{-\frac{8}{m^2}-\frac{16}{m^3}} \\
\frac{16}{m^3}
\end{array}$$

3. $a+b)\,a-b\,(1-\frac{2b}{a}+\frac{2b^2}{a^2}-\frac{2b^3}{a^3}+\frac{2b^4}{a^4}\cdots$

$$\begin{array}{l}
\underline{a+b} \\
-2b \\
\underline{-2b-\frac{2b^2}{a}} \\
\frac{2b^2}{a} \\
\underline{\frac{2b^2}{a}+\frac{2b^3}{a^2}} \\
-\frac{2b^3}{a^2} \\
\underline{-\frac{2b^3}{a^2}-\frac{2b^4}{a^3}} \\
\frac{2b^4}{a^3}
\end{array}$$

4. $a^2-x^2)\,a^2+x^2\,(1+\frac{2x^2}{a^2}+\frac{2x^4}{a^4}+\frac{2x^6}{a^6}+\frac{2x^8}{a^8}\cdots$

$$\begin{array}{l}
\underline{a^2-x^2} \\
2x^2 \\
\underline{2x^2-\frac{2x^4}{a^2}} \\
\frac{2x^4}{a^2} \\
\underline{\frac{2x^4}{a^2}-\frac{2x^6}{a^4}} \\
\frac{2x^6}{a^4} \\
\underline{\frac{2x^6}{a^4}-\frac{2x^8}{a^6}} \\
\frac{2x^8}{a^6}
\end{array}$$

5. $a-x)\,ax\,(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\frac{x^5}{a^4}\cdots$

$$\begin{array}{l}
\underline{ax-x^2} \\
x^2 \\
\underline{x^2-\frac{x^3}{a}} \\
\frac{x^3}{a} \\
\underline{\frac{x^3}{a}-\frac{x^4}{a^2}} \\
\frac{x^4}{a^2} \\
\underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}} \\
\frac{x^5}{a^3}
\end{array}$$

6. $a+x)\,b\,(\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}\ldots$

$\underline{b+\frac{bx}{a}}$

$-\frac{bx}{a}$

$\underline{-\frac{bx}{a}-\frac{bx^2}{a^2}}$

$\frac{bx^2}{a^2}$

$\underline{\frac{bx^2}{a^2}-\frac{bx^3}{a^3}}$

$\frac{bx^3}{a^3}$

$\underline{\frac{bx^3}{a^3}-\frac{bx^4}{a^4}}$

$\frac{bx^4}{a^4}$

7. $1+2x-2x^2)\,1\,(1-2x+6x^2-16x^3+44x^4\ldots$

$\underline{1+2x-2x^2}$

$-2x+2x^2$

$\underline{-2x-4x^2+4x^3}$

$6x^2-4x^3$

$\underline{6x^2+12x^3-12x^4}$

$-16x^3+12x^4$

$\underline{-16x^3-32x^4+32x^5}$

$-44x^4-32x^5$

8. $1-x+x^2)\,1+x\,(1+2x+x^2-x^3-2x^4\ldots$

$\underline{1-x+x^2}$

$2x-x^2$

$\underline{2x-2x^2+2x^3}$

x^2-2x^3

$\underline{x^2-x^3+x^4}$

$-x^3-x^4$

$\underline{-x^3+x^4-x^5}$

$-2x^4+x^5$

9. $1-2b)\,1+b\,(1+3b+6b^2+12b^3+24b^4\ldots$

$\underline{1-2b}$

$3b$

$\underline{3b-6b^2}$

$6b^2$

$\underline{6b^2-12b^3}$

$12b^3$

$\underline{12b^3-24b^4}$

$24b^4$

10. $$\begin{array}{l}
x+b)\,x^3-b^3\,\left(x^2-bx+b^2-\dfrac{2b^3}{x}+\dfrac{2b^4}{x^2}\ldots\right.\\
\underline{x^3+bx^2}\\
-bx^2-b^3\\
\underline{-bx^2-b^2x}\\
b^2x-b^3\\
\underline{b^2x+b^3}\\
-2b^3\\
\underline{-2b^3-\dfrac{2b^4}{x}}\\
\dfrac{2b^4}{x}
\end{array}$$

11. $$\begin{array}{l}
x-b)\,a^2\,\left(\dfrac{a^2}{x}+\dfrac{a^2b}{x^2}+\dfrac{a^2b^2}{x^3}+\dfrac{a^2b^3}{x^4}+\dfrac{a^2b^4}{x^5}\ldots\right.\\
\underline{a^2-\dfrac{a^2b}{x}}\\
\dfrac{a^2b}{x}\\
\underline{\dfrac{a^2b}{x}-\dfrac{a^2b^2}{x^2}}\\
\dfrac{a^2b^2}{x^2}\\
\underline{\dfrac{a^2b^2}{x^2}-\dfrac{a^2b^3}{x^3}}\\
\dfrac{a^2b^3}{x^3}\\
\underline{\dfrac{a^2b^3}{x^3}-\dfrac{a^2b^4}{x^4}}\\
\dfrac{a^2b^4}{x^4}
\end{array}$$

12. $$\begin{array}{l}
a^2+2ax+x^2)\,a^2\,\left(1-\dfrac{2x}{a}+\dfrac{3x^2}{a^2}-\dfrac{4x^3}{a^3}+\dfrac{5x^4}{a^4}\ldots\right.\\
\underline{a^2+2ax+x^2}\\
-2ax-x^2\\
\underline{-2ax-4x^2-\dfrac{2x^3}{a}}\\
3x^2+\dfrac{2x^3}{a}\\
\underline{3x^2+\dfrac{6x^3}{a}+\dfrac{3x^4}{a^2}}\\
-\dfrac{4x^3}{a}-\dfrac{3x^4}{a^2}\\
\underline{-\dfrac{4x^3}{a}-\dfrac{8x^4}{a^2}-\dfrac{4x^5}{a^3}}\\
\dfrac{5x^4}{a^2}+\dfrac{4x^5}{a^3}
\end{array}$$

13. Dividend $=$(Divisor and Quotient) + Remainder

$$=(x-a)(x^2-2ax)+4a^3=x^3-3ax^2+2a^2x+4a^3.$$

14. Dividend $=(m-5)(m^3+5m^2+15m+34)+75$

$$=m^4-10m^2-41m-95.$$

LXIV.

1. $\dfrac{10x^2+15x+6}{30}\times\dfrac{4x+3}{12}=\dfrac{40x^3+90x^2+69x+18}{360}$, etc.

2. $\dfrac{6a^2-5a+10}{30}\times\dfrac{5a-4}{20}=\dfrac{30a^3-49a^2+70a-40}{600}$, etc.

3.
$$\begin{array}{l}
x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3} \\
x-\dfrac{1}{x} \\
\hline
x^4+x^2+1+\dfrac{1}{x^2} \\
\quad -x^2-1-\dfrac{1}{x^2}-\dfrac{1}{x^4} \\
\hline
x^4 \qquad\qquad\qquad -\dfrac{1}{x^4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+\dfrac{1}{x^2} \\
x^2+1+\dfrac{1}{x^2} \\
\hline
x^4-x^2+1 \\
\quad +x^2-1+\dfrac{1}{x^2} \\
\qquad\quad +1-\dfrac{1}{x^2}+\dfrac{1}{x^4} \\
\hline
x^4 \quad +1 \qquad +\dfrac{1}{x^4}
\end{array}$$

5. $\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)=\dfrac{1}{a^4}-\dfrac{1}{b^4}.$

6. $\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{ab}-\dfrac{1}{b^2}+\dfrac{1}{bc}+\dfrac{1}{ac}-\dfrac{1}{bc}+\dfrac{1}{c^2}=\dfrac{1}{a^2}+\dfrac{2}{ac}-\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

7. $1+\dfrac{b}{a}+\dfrac{b^2}{a^2}-\dfrac{b}{a}-\dfrac{b^2}{a^2}-\dfrac{b^3}{a^3}+\dfrac{b^2}{a^2}+\dfrac{b^3}{a^3}+\dfrac{b^4}{a^4}=1+\dfrac{b^2}{a^2}+\dfrac{b^4}{a^4}.$

8.
$$\begin{array}{rrrrrr}
1 & -\frac{1}{2}x & +\frac{1}{8}x^2 & -\frac{1}{16}x^3 & & \\
1 & +\frac{1}{2}x & +\frac{1}{4}x^2 & & & \\
\hline
1 & -\frac{1}{2}x & +\frac{1}{8}x^2 & -\frac{1}{16}x^3 & & \\
 & +\frac{1}{2}x & -\frac{1}{4}x^2 & +\frac{1}{16}x^3 & -\frac{1}{32}x^4 & \\
 & & +\frac{1}{4}x^2 & -\frac{1}{8}x^3 & +\frac{1}{32}x^4 & -\frac{1}{64}x^5 \\
\hline
1 & +\frac{x^2}{8} & & -\frac{x^3}{8} & -\frac{x^5}{64}. &
\end{array}$$

9. $\frac{15+18x-14x^2}{6x^2} \times \frac{4-2x-x^2}{2x^2} = \frac{60+42x-107x^2+10x^3+14x^4}{12x^4}$, etc.

10. $\frac{a^4}{b^4}+1+\frac{2a^2}{b^2}-1-\frac{b^4}{a^4}-\frac{2b^2}{a^2}-\frac{2a^2}{b^2}-\frac{2b^2}{a^2}-4=\frac{a^4}{b^4}-\frac{b^4}{a^4}-\frac{4b^2}{a^2}-4.$

LXV.

1. $\frac{x^4-1}{x^2} \div \frac{x^2+1}{x} = \frac{x^2-1}{x}$, etc.

2. $\frac{a^2b^2-1}{b^2} \div \frac{ab-1}{b} = \frac{ab+1}{b}$, etc.

3. $\frac{m^3n^3+1}{n^3} \div \frac{mn+1}{n} = \frac{m^2n^2-mn+1}{n^2}$, etc.

4. $\frac{c^5d^5-1}{d^5} \div \frac{cd-1}{d} = \frac{c^4d^4+c^3d^3+c^2d^2+cd+1}{d^4}$, etc.

5. $\frac{x^4+2x^2y^2+y^4}{x^2y^2} \div \frac{x^2+y^2}{xy} = \frac{x^2+y^2}{xy}$, etc.

6. $\frac{b^4+a^2b^2+a^4}{a^4b^4} \div \frac{b^2-ab+a^2}{a^2b^2} = \frac{b^2+ab+a^2}{a^2b^2}$, etc.

7. $$\frac{x^6-y^6-3x^4y^2+3x^2y^4}{x^3y^3}\div\frac{x^2-y^2}{xy}=\frac{x^4-2x^2y^2+y^4}{x^2y^2}, \text{ etc.}$$

8. $$\frac{6x^5-32x^4+77x^3-86x^2-66x+216}{8}\div\frac{x^2-2x+6}{2}$$
$$=\frac{6x^3-10x^2+16x+36}{4}, \text{ etc.}$$

9. $$\frac{a^6+b^6}{a^3b^3}\div\frac{a^2+b^2}{ab}=\frac{a^4-a^2b^2+b^4}{a^2b^2}, \text{ etc.}$$

10. $$\frac{b^3c^3+a^3c^3+a^3b^3-3a^2b^2c^2}{a^3b^3c^3}\div\frac{bc+ac+ab}{abc}$$
$$=\frac{b^2c^2-abc^2-ab^2c+a^2c^2-a^2bc+a^2b^2}{a^2b^2c^2}, \text{ etc.}$$

LXVI.

1. $$\frac{x-3}{10}\times\frac{50x+7}{100}=\frac{50x^2-143x-21}{1000}, \text{ etc.}$$

2. $$\left(\frac{5x}{100}+7\right)\times\left(\frac{2x}{10}-3\right)=\frac{5x+700}{100}\times\frac{2x-30}{10}=\frac{10x^2+1250x-21000}{1000}, \text{ etc.}$$

3. $$\frac{3x-2y}{10}\times\frac{4x+7y}{10}=\frac{12x^2+13xy-14y^2}{100}, \text{ etc.}$$

4. $$\frac{43x+52y}{10}\times\frac{4x-6y}{100}=\frac{172x^2-50xy-312y^2}{1000}, \text{ etc.}$$

5. ·000027 − ·001 + ·000343 + ·00063 = ·001 − ·001 = 0.

6. ·343 − ·0441 + ·00189 − ·000027 = ·34489 − ·044127 = ·300763.

LXVII.

1. $$a_1x\left(\frac{a_1x+a_2x^2+a_3x^3+a_4x^4+\dots}{a_1x}\right)=a\,x\left(1+\frac{a_2x}{a_1}+\frac{a_3x^2}{a_1}+\frac{a_4x^3}{a_1}+\dots\right).$$

2. $xyz\left(\dfrac{xy-xz+yz}{xyz}\right)=xyz\left(\dfrac{1}{z}-\dfrac{1}{y}+\dfrac{1}{x}\right).$

3. $x^2\left(\dfrac{x^2+xy+y^2}{x^2}\right)=x^2\left(1+\dfrac{y}{x}+\dfrac{y^2}{x^2}\right).$

4. $$(a+b)\left\{\frac{(a+b)^3-c(a+b)^2-d(a+b)+e}{a+b}\right\}$$

$$=(a+b)\left\{(a+b)^2-c(a+b)-d+\frac{e}{a+b}\right\}.$$

LXVIII.

1. Let $\dfrac{a}{b}=\lambda$; then $\dfrac{c}{d}=\lambda$; and $a=\lambda b$, $c=\lambda d$. Then

(1.) $\dfrac{a-b}{b}=\dfrac{\lambda b-b}{b}=\lambda-1$

$\dfrac{c-d}{d}=\dfrac{\lambda d-d}{d}=\lambda-1.$

(2.) $\dfrac{a}{a+b}=\dfrac{\lambda b}{\lambda b+b}=\dfrac{\lambda}{\lambda+1}$

$\dfrac{c}{c+d}=\dfrac{\lambda d}{\lambda d+d}=\dfrac{\lambda}{\lambda+1}.$

(3.) $\dfrac{3a}{4a-5b}=\dfrac{3\lambda b}{4\lambda b-5b}=\dfrac{3\lambda}{4\lambda-5}$

$\dfrac{3c}{4c-5d}=\dfrac{3\lambda d}{4\lambda d-5d}=\dfrac{3\lambda}{4\lambda-5}.$

(4.) $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{\lambda^2b^2+b^2}{\lambda^2b^2-b^2}=\dfrac{\lambda^2+1}{\lambda^2-1}$

$\dfrac{c^2+d^2}{c^2-d^2}=\dfrac{\lambda^2d^2+d^2}{\lambda^2d^2-d^2}=\dfrac{\lambda^2+1}{\lambda^2-1}.$

(5.) $\dfrac{8a+b}{4a+7b}=\dfrac{8\lambda b+b}{4\lambda b+7b}=\dfrac{8\lambda+1}{4\lambda+7}$

$\dfrac{8c+d}{4c+7d}=\dfrac{8\lambda d+d}{4\lambda d+7d}=\dfrac{8\lambda+1}{4\lambda+7}.$

(6.) $$\frac{a^2-b^2}{c^2-d^2}=\frac{\lambda^2b^2-b^2}{\lambda^2d^2-d^2}=\frac{b^2(\lambda^2-1)}{d^2(\lambda^2-1)}=\frac{b^2}{d^2}$$

$$\frac{ab}{cd}=\frac{\lambda b\times b}{\lambda d\times d}=\frac{b^2}{d^2}.$$

(7.) $$\frac{11a+b}{11c+d}=\frac{11\lambda b+b}{11\lambda d+d}=\frac{b(11\lambda+1)}{d(11\lambda+1)}=\frac{b}{d}$$

$$\frac{13a+b}{13c+d}=\frac{13\lambda b+b}{13\lambda d+d}=\frac{b(13\lambda+1)}{d(13\lambda+1)}=\frac{b}{d}.$$

(8.) $$\frac{a^2-ab+b^2}{a^2+ab+b^2}=\frac{\lambda^2b^2-\lambda b^2+b^2}{\lambda^2b^2+\lambda b^2+b^2}=\frac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}$$

$$\frac{c^2-cd+d^2}{c^2+cd+d^2}=\frac{\lambda^2d^2-\lambda d^2+d^2}{\lambda^2d^2+\lambda d^2+d^2}=\frac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}.$$

2. Let each of the fractions $=\lambda$

Then $l=a\lambda-b\lambda$; $m=b\lambda-c\lambda$; $n=c\lambda-a\lambda$

And $l+m+n=a\lambda-b\lambda+b\lambda-c\lambda+c\lambda-a\lambda=0$.

3. Let each of the fractions $=\lambda$

Then $a=\lambda b$, $c=\lambda d$, $e=\lambda f$

And $\dfrac{la+mc+ne}{lb+md+nf}=\dfrac{l\lambda b+m\lambda d+n\lambda f}{lb+md+nf}=\lambda=\dfrac{a}{b}$.

4. Let each of the fractions $=\lambda$

Then $a+b=\lambda b$
$b+c=\lambda c$
$c+a=\lambda a$

Adding, $2a+2b+2c=\lambda(a+b+c)$

Hence $\lambda=2$, and $\therefore a+b=2b$, or $a=b$

Similarly we can show that $a=c$, and $b=c$.

5. Let each of the fractions $=\lambda$

Then $a_1=\lambda b_1$; $a_2=\lambda b_2$; $a_3=\lambda b_3$

And $\dfrac{2a_1+3a_2+4a_3}{2b_1+3b_2+4b_3}=\dfrac{2\lambda b_1+3\lambda b_2+4\lambda b_3}{2b_1+3b_2+4b_3}=\lambda=\dfrac{a_1}{b_1}$.

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}$ is less than λ ; and $\frac{e}{f}$ is less than λ

$\therefore$ $a=\lambda b$; c is less than λd ; e is less than λf

$\therefore$ $a+c+e$ is less than $\lambda b+\lambda d+\lambda f$

$\therefore$ $\frac{a+c+e}{b+d+f}$ is less than λ, that is, less than $\frac{a}{b}$

Next let $\frac{e}{f}=\mu$; then $\frac{a}{b}$ and $\frac{c}{d}$ are each greater than μ

$\therefore$ $e=\mu f$; a is greater than μb ; c is greater than μd

$\therefore$ $a+c+e$ is greater than $\mu b+\mu d+\mu f$

$\therefore$ $\frac{a+c+e}{b+d+f}$ is greater than μ, that is than $\frac{e}{f}$.

7. Let each fraction $=\lambda$; then $x_1=\lambda y_1$; and $x_2=\lambda y_2$

Then $\frac{4x_1+5y_1}{7x_1+9y_1}=\frac{4\lambda y_1+5y_1}{7\lambda y_1+9y_1}=\frac{4\lambda+5}{7\lambda+9}$

And $\frac{4x_2+5y_2}{7x_2+9y_2}=\frac{4\lambda y_2+5y_2}{7\lambda y_2+9y_2}=\frac{4\lambda+5}{7\lambda+9}$.

8. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{a^2+ab}{c^2+cd}=\frac{\lambda^2b^2+\lambda b^2}{\lambda^2d^2+\lambda d^2}=\frac{b^2(\lambda^2+\lambda)}{d^2(\lambda^2+\lambda)}=\frac{b^2}{d^2}$

And $\frac{ab-b^2}{cd-d^2}=\frac{\lambda b^2-b^2}{\lambda d^2-d^2}=\frac{b^2(\lambda-1)}{d^2(\lambda-1)}=\frac{b^2}{d^2}$.

9. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{7a+b}{3a+5b}=\frac{7\lambda b+b}{3\lambda b+5b}=\frac{7\lambda+1}{3\lambda+5}$

And $\frac{7c+d}{3c+5d}=\frac{7\lambda d+d}{3\lambda d+5d}=\frac{7\lambda+1}{3\lambda+5}$.

10. Since b is greater than a

bc is greater than ac

$ab+bc$ is greater than $ab+ac$

$\frac{ab+bc}{b(b+c)}$ is greater than $\frac{ab+ac}{b(b+c)}$

$\frac{b(a+c)}{b(b+c)}$ is greater than $\frac{a(b+c)}{b(b+c)}$

$\frac{a+c}{b+c}$ is greater than $\frac{a}{b}$.

11. Since b is less than a, we can show, as in the preceding solution that $\frac{a+c}{b+c}$ is less than $\frac{a}{b}$.

LXIX.

1. $3\times16+\dfrac{2\times4\times\frac{1}{4}}{1}-\dfrac{1}{\frac{1}{4}}=48+2-4=46.$

2.
$$\begin{array}{r}
7x^2-12x+5)\,14x^3+7x^2-56x+35\,(2x\\
\underline{14x^3-24x^2+10x}\phantom{{}+35\,(2x}\\
31x^2-66x+35\\
7\\
\hline
217x^2-462x+245\,(31\\
\underline{217x^2-372x+155}\\
-90x+90
\end{array}$$

Hence H. C. F. is $x-1$, and dividing both terms of the first fraction by it, we get $\frac{2x^2+3x-5}{7x-5}$

$$\begin{array}{r}
a^2+4a-45)\,a^3-39a+70\,(a-4\\
\underline{a^3+4a^2-45a}\\
-4a^2+6a+70\\
\underline{-4a^2-16a+180}\\
22a-110
\end{array}$$

Hence we find $a-5$ to be the H. C. F., and dividing both terms of the fraction by it, we get $\frac{a^2+5a-14}{a+9}$.

3. $\dfrac{a^2+2ap+p^2-a^2+2ap-p^2}{a^2-p^2}\div\dfrac{a^2+2ap+p^2+a^2-2ap+p^2}{a^2-p^2}$

$$=\frac{4ap}{2a^2+2p^2}=\frac{2ap}{a^2+p^2}.$$

4. $$\frac{6x^2-4y^2+3z^2}{24}+\frac{6y^2-4z^2+3x^2}{24}+\frac{6z^2+4x^2+3y^2}{24}=\frac{13x^2+5y^2+5z^2}{24}$$

$$\frac{13x^2+5y^2+5z^2}{24}-\frac{24z^2-24x^2+12y^2}{24}=\frac{37x^2-7y^2-19z^2}{24}.$$

5. $$\frac{16+\frac{1}{4}-1+4}{16-\frac{1}{4}-1+1}=\frac{19\frac{1}{4}}{15\frac{3}{4}}=\frac{77}{63}=\frac{11}{9}.$$

6. $$\frac{15x^2+18ax-14a^2}{6}\times\frac{4x^2-2ax-a^2}{2}$$

$$=\frac{60x^4+42ax^3-107a^2x^2+10a^3x+14a^4}{12}.$$

7. $$\frac{(a-b)(a^2+ab+b^2)}{(a-b)^2}=\frac{a^2+ab+b^2}{a-b}=a+2b+\frac{3b^2}{a-b},$$ by division.

8. $$\frac{x^2-2xy+y^2+2xy}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}=\frac{x^2+y^2}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}$$

$$=\frac{x^3+xy^2+x^2y+y^3+y^3-xy^2}{x(x^2-y^2)}=\frac{x^3+x^2y+2y^3}{x(x^2-y^2)}.$$

9. $$\begin{array}{l}
5x^2+9x-2)\,60x^3-17x^2-4x+1\,(12x-25\\
\qquad\qquad\quad\underline{60x^3+108x^2-24x}\\
\qquad\qquad\qquad\quad -125x^2+20x+1\\
\qquad\qquad\qquad\quad \underline{-125x^2-225x+50}\\
\qquad\qquad\qquad\qquad\qquad 245x-49
\end{array}$$

and $$\frac{245x-49}{5x^2+9x-2}=\frac{49(5x-1)}{(x+2)(5x-1)}=\frac{49}{x+2}.$$

10. $$\begin{array}{l}
x^4-9x^3+7x^2+9x-8)\,x^4+7x^3-9x^2-7x+8\,(1\\
\qquad\qquad\qquad\qquad\quad \underline{x^4-9x^3+7x^2+9x-8}\\
\qquad\qquad\qquad\qquad\qquad 16x^3-16x^2-16x+16
\end{array}$$

Divide by 16; $x^3-x^2-x+1)\,x^4-9x^3+7x^2+9x-8\,(x-8$

$$\begin{array}{r} x^4-x^3-x^2+x \\ \hline -8x^3+8x^2+8x-8 \\ -8x^3+8x^2+8x-8 \\ \hline \end{array}$$

Hence x^3-x^2-x+1 is the H. C. F.

11. $$\frac{x^3}{x^4-1}+\frac{1}{1-x-\dfrac{1-x}{1-x^2-1}}=\frac{x^3}{x^4-1}+\frac{-x^2}{-x^2+x^3-1+x}$$

$$=\frac{x^3}{x^4-1}+\frac{-x^2(x+1)}{x^4-1}=\frac{-x^2}{x^4-1}=\frac{x^2}{1-x^4}.$$

12. $$a+ab+b^2\left(a+ab+\frac{ab^2}{1-b}\right)=a+ab+b^2\left(\frac{a+ab-ab-ab^2+ab^2}{1-b}\right)$$

$$=a+ab+\frac{ab^2}{1-b}=\frac{a+ab-ab-ab^2+ab^2}{1-b}=\frac{a}{1-b}.$$

13. $$\left(l+\frac{1}{l}\right)\left(l-\frac{1}{l}\right)\left(l^2+\frac{1}{l^2}\right)=\left(l^2-\frac{1}{l^2}\right)\left(l^2+\frac{1}{l^2}\right)=l^4-\frac{1}{l^4}.$$

14. $$\frac{a+b+2}{ab+a+b+1}+\frac{1}{c+1}=\frac{ac+bc+2c+a+b+2+ab+a+b+1}{abc+ab+ac+a+bc+b+c+1}$$

$$=\frac{ab+ac+bc+2a+2b+2c+3}{abc+ab+ac+a+bc+b+c+1}$$

If this fraction $=1$, then $ab+ac+bc+2a+2b+2c+3$

$=abc+ab+ac+a+bc+b+c+1$;

or, $ab+ac+bc+2a+2b+2c+3-ab-ac-a-bc-b-c-1=abc$,

or, $a+b+c+2=abc$.

15. $\frac{ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2}{a^2x^2} \div x - a$

$$x - a)\ ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2\ (ax^2 - abx - b^2x - ab^2$$

$$\underline{ax^3 - a^2x^2}$$

$$-abx^2 - b^2x^2 + a^2bx$$

$$\underline{-abx^2 + a^2bx}$$

$$-b^2x^2 + a^2b^2$$

$$\underline{-b^2x^2 + ab^2x}$$

$$-ab^2x + a^2b^2$$

$$\underline{-ab^2x + a^2b^2}$$

Hence the result is $\frac{ax^2 - abx - b^2x - ab^2}{a^2x^2}$, or, $\frac{1}{a} - \frac{b}{ax} - \frac{b^2}{a^2x} - \frac{b^2}{ax^2}$.

16. $\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) \div \left(\frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}\right) = 1$

Also, since $s = \frac{a+b+c}{2}$

$s - a = \frac{b+c-a}{2}$; $s - b = \frac{a+c-b}{2}$; $s - c = \frac{a+b-c}{2}$.

Hence $\frac{s.(s-a)\ +\ (s-b)\ (s-c)}{bc} = \frac{\frac{(b+c)^2 - a^2}{4} + \frac{a^2 - (b-c)^2}{4}}{bc}$

$$= \frac{b^2 + 2bc + c^2 - a^2 + a^2 - b^2 + 2bc - c^2}{4bc} = \frac{4bc}{4bc} = 1.$$

17. $\frac{1}{1 + \frac{x}{a}} + \frac{1}{1 - \frac{x}{a}} + \frac{2}{1 + \frac{x^2}{a^2}} = \frac{a}{a+x} + \frac{a}{a-x} + \frac{2a^2}{a^2+x^2}$

$$= \frac{a^2 - ax + a^2 + ax}{a^2 - x^2} + \frac{2a^2}{a^2 + x^2} = \frac{2a^2(a^2 + x^2)\ +\ 2a^2\,(a^2 - x^2)}{a^4 - x^4} = \frac{4a^4}{a^4 - x^4}$$

18. $$\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}=\frac{2(a^2+b^2)}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}$$

$$=\frac{2(a^4+2a^2b^2+b^4)-2(a^4-2a^2b^2+b^4)}{a^4-b^4}=\frac{8a^2b^2}{a^4-b^4}.$$

19. $$\frac{2ab-a^2-2ab-b^2}{2a(a+b)}+\frac{a^2+b^2}{2a(a-b)}=\frac{(a-b)(-a^2-b^2)+(a+b)(a^2+b^2)}{2a(a^2-b^2)}$$

$$=\frac{-a^3-ab^2+a^2b+b^3+a^3+ab^2+a^2b+b^3}{2a(a^2-b^2)}=\frac{2a^2b+2b^3}{2a(a^2-b^2)}=\frac{b(a^2+b^2)}{a(a^2-b^2)}.$$

20. $$\frac{a^2-ab+b^2}{(a-b)^3}\times\frac{(a+b)(a-b)}{a^2+b^2}=\frac{(a^2-ab+b^2)(a+b)}{(a-b)^2(a^2+b^2)}=\frac{a^3+b^3}{(a-b)^2(a^2+b^2)}.$$

21. $$\frac{2}{(x^2-1)^2}-\frac{1}{2(x-1)^2}+\frac{1}{x^2-1}=\frac{2}{(x^2-1)^2}-\frac{(x+1)^2}{2(x^2-1)^2}+\frac{x^2-1}{(x^2-1)^2}$$

$$=\frac{4-x^2-2x-1+2x^2-2}{2(x^2-1)^2}=\frac{x^2-2x+1}{2(x^2-1)^2}=\frac{1}{2(x+1)^2}.$$

22. $$\frac{(a+b)^2-c^2}{c^2-(a-b)^2}\div\frac{a+b+c}{b+c-a}=\frac{(a+b+c)(a+b-c)(b+c-a)}{(c+a-b)(c-a+b)(a+b+c)}=\frac{a+b-c}{a-b+c}.$$

23. $$\left(\frac{x^2}{x+1}+1-\frac{1}{x+1}\right)\div\left(\frac{x^2}{x-1}-x-\frac{1}{x-1}\right)$$

$$=\frac{x^2+x+1-1}{x+1}\div\frac{x^2-x^2+x-1}{x-1}=\frac{x^2+x}{x+1}\div 1=\frac{x(x+1)}{x+1}=x.$$

24. $$\left(\frac{\frac{a+b}{2}-a}{\frac{a+b}{2}-b}\right)^3-\frac{\frac{a+b}{2}-2a+b}{\frac{a+b}{2}+a-2b}=\left(\frac{b-a}{a-b}\right)^3-\frac{3b-3a}{3a-3b}=(-1)^3-(-1)$$

$$=-1+1=0.$$

25. $$\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$

$$=\frac{a+b-c}{a+c+b}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{b+c+a}=\frac{b+c+a}{a+b+c}=1.$$

26. $$\frac{x(x-4)(x^2-4)(x^2-4)}{x^2(x-2)(x-2)}=\frac{(x-4)(x+2)(x+2)}{x}=\frac{(x-4)(x+2)^2}{x}.$$

27. $$\frac{(a+1)(a-1)(a^3+1)(a^3-1)}{(a+1)(a+1)a^2(a-1)(a-1)}=\frac{(a^2-a+1)(a^2+a+1)}{a^2}=\frac{a^4+a^2+1}{a^2}.$$

28. $$\frac{1+x-x^2}{x^3}-\frac{1}{(x^2+1)^2}+\frac{(x^2+1)(x-1)}{(x^2+1)^2}-\frac{3}{x^2(x^2+1)^2}$$
$$=\frac{1+x-x^2}{x^3}-\frac{x^3}{x^3(x^2+1)^2}+\frac{x^3(x^3-x^2+x-1)}{x^3(x^2+1)^2}-\frac{3x}{x^3(x^2+1)^2}$$
$$=\frac{(x^4+2x^2+1)(1+x-x^2)-x^3+x^3(x^3-x^2+x-1)-3x}{x^3(x^2+1)^2}$$
$$=\frac{1+x+x^2+2x^3-x^4+x^5-x^6-x^3+x^6-x^5+x^4-x^3-3x}{x^3(x^2+1)^2}$$
$$=\frac{1-2x+x^2}{x^3(x^2+1)^2}=\frac{(x-1)^2}{x^3(x^2+1)^2}.$$

29. $$\frac{x^6-a^2x^4+a^4x^2-a^6}{a^3x^3}\div\frac{x^2-a^2}{ax}=\frac{x^4+a^4}{a^2x^2}=\frac{x^2}{a^2}+\frac{a^2}{x^2}.$$

30. $$\left\{\frac{a^2+2ab+b^2-a^2+2ab-b^2}{2(a^2-b^2)}+\frac{2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}$$
$$=\left\{\frac{2ab+2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}=\frac{2b(a+b)(a-b)}{(a+b)(a-b)2b}=1.$$

31. $$\frac{a^2+b^2+c^2+2ab+2ac+2bc+b^2-2bc+c^2+c^2-2ac+a^2-2ab+b^2}{a^2+b^2+c^2}$$
$$=\frac{3a^2+3b^2+3c^2}{a^2+b^2+c^2}=3.$$

32. $$\frac{1+3x^2+2x^3}{(3-2x-7x^2)^4}-\frac{(1-x-3x^2)(3-2x-7x^2)}{(3-2x-7x^2)^4}$$
$$=\frac{1+3x^2+2x^3-(3-5x-14x^2+13x^3+21x^4)}{(3-2x-7x^2)^4}$$
$$=\frac{-2+5x+17x^2-11x^3-21x^4}{(3-2x-7x^2)^4}.$$

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33. $$\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{(x^2-y^2)(x^2+y^2)} \div \frac{x^2+2xy+y^2-x^2+2xy-y^2}{(x-y)(x+y)}$$
$$=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)} \times \frac{(x-y)(x+y)}{4xy}=\frac{xy}{x^2+y^2}.$$

34. $$\frac{x^2-y^2}{y^2} \times \frac{y}{x-y}+\frac{x^3-y^3}{y^3} \times \frac{-y^2}{x^2+xy+y^2}=\frac{x+y}{y}+\frac{(x-y)(-1)}{y}=\frac{2y}{y}=2.$$

35. $$\frac{a(a-b)}{(a-b)(a^2+ab+b^2)} \times \frac{a^2+ab+b^2}{a+b}+\frac{a^3-b^3}{a^3+b^3} \times \frac{a^2-ab+b^2}{a^2+ab+b^2}$$
$$=\frac{a}{a+b}+\frac{a-b}{a+b}=\frac{2a-b}{a+b}.$$

36. $$\frac{2-(x-1)}{4(x-1)^2}+\frac{(x-1)^2-4}{4(x-1)^2(x+1)}=\frac{3-x}{4(x-1)^2}+\frac{x^2-2x-3}{4(x-1)^2(x+1)}$$
$$=\frac{3x+3-x^2-x+x^2-2x-3}{4(x-1)^2(x+1)}=0.$$

37. $$\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(a-b)(x-a)+bx}{abx(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(ax+ab-a^2)(x-b)-ax(x-a)}{abx(a-b)(x-a)(x-b)}$$
$$=\frac{ax^2-abx+abx-ab^2-a^2x+a^2b-ax^2+a^2x}{abx(a-b)(x-c)(x-b)}=\frac{1}{x(x-a)(x-b)}.$$

38. $$\frac{s}{a}-\frac{a}{a}+\frac{s}{b}-\frac{b}{b}+\frac{s}{c}-\frac{c}{c}+\ldots\text{to } n \text{ terms}$$
$$=\frac{s}{a}+\frac{s}{b}+\frac{s}{c}+\ldots\text{ to } n \text{ terms}-(1+1+1\ldots\text{ to } n \text{ terms})$$
$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right)-n.$$

39. $$\frac{x^4+x^2y^2-x^2y^2+y^4}{x^4-y^4} \times \frac{(x^2-y^2)^2}{x^4-2x^2y^2+y^4+x^4+2x^2y^2+y^4}$$
$$=\frac{(x^4+y^4)(x^2-y^2)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)2(x^4+y^4)}=\frac{x^2-y^2}{2(x^2+y^2)}.$$

40. $\frac{a+x+a-x}{a+x-a+x} \div \frac{a^2+x^2+a^2-x^2}{a^2+x^2-a^2+x^2} = \frac{2a}{2x} \times \frac{2x^2}{2a^2} = \frac{x}{a}$.

41. Since $x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$; and $\frac{1}{x^2}-x^2 = \left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)$

the quotient is $x^2-1+\frac{1}{x^2}-3\left(\frac{1}{x}-x\right)+4$;

or, $x^2+3x+3-\frac{3}{x}+\frac{1}{x^2}$.

The quotient may also be obtained by long division as in p. 133 of the *Algebra*.

42. $\left(\frac{s}{s}-\frac{a}{s}\right)+\left(\frac{s}{s}-\frac{b}{s}\right)+\left(\frac{s}{s}-\frac{c}{s}\right)+\ldots$ to n terms

$=(1+1+1\ldots$ to n terms$)-\frac{1}{s}(a+b+c+\ldots$ to n terms$)$

$=n-\frac{s}{s}=n-1$.

43. $\frac{x^2+xy-xy+y^2}{x^2-y^2} \div \frac{x^4-x^2y^2+x^2y^2+y^4}{(x^2+y^2)(x^2-y^2)} = \frac{x^2+y^2}{x^2-y^2} \times \frac{(x^2+y^2)(x^2-y^2)}{x^4+y^4}$

$=\frac{(x^2+y^2)^2}{x^4+y^4}$.

44. $$\frac{\dfrac{x^2+2xy+y^2-2xy}{(x+y)^2}}{\dfrac{x^2-2xy+y^2+2xy}{(x-y)^2}} + \frac{(x-y)^2}{(x+y)^2} = \frac{(x^2+y^2)(x-y)^2}{(x+y)^2(x^2+y^2)} \times \frac{(x+y)^2}{(x-y)^2} = 1.$$

45. $(a+b)(cd-1)=(1-ab)(c+d)$, or $acd-a+bcd-b=c-abc+d-abd$,

or $a+b+c+d=bcd+acd+abd+abc$

$=abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

$\therefore \frac{a+b+c+d}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=abcd$.

46. $\frac{(p+q)^4}{(p-q)^4} \div \frac{(p+q)^3}{(p-q)^3} = \frac{p+q}{p-q}$.

47. $\dfrac{1-2x}{3(x^2-x+1)}+\dfrac{1}{6(x+1)}+\dfrac{x+1}{2(x^2+1)}$

$$=\frac{(1-2x)(2x+2)+(x^2-x+1)}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{2x+2-4x^2-4x+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{3-3x-3x^2}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{x^2-x^3-x^4+1-x-x^2+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

48. $\dfrac{1}{x+\dfrac{z}{yz+1}}\div\dfrac{y}{xy+1}-\dfrac{1}{y(xyz+x+z)}$

$$=\frac{yz+1}{xyz+x+z}\times\frac{xy+1}{y}-\frac{1}{y(xyz+x+z)}=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}=1.$$

49. $\dfrac{1}{a-x}+\dfrac{x}{(a-x)^2}-\left(\dfrac{1}{a-y}+\dfrac{y}{(a-y)^2}\right)=\dfrac{a}{(a-x)^2}-\dfrac{a}{(a-y)^2}$

And $\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-y)^2(a-x)}=\dfrac{(a-y)-(a-x)}{(a-x)^2(a-y)^2}$

Hence original fraction $=\dfrac{a(a-y)^2-a(a-x)^2}{(a-y)-(a-x)}=a(a-y)+a(a-x)$

$$=2a^2-ax-ay.$$

50. $\dfrac{3}{abc}\div\dfrac{a+b-c}{abc}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{3}{a+b-c}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{a+b+c}{a+b-c}.$

51. $\dfrac{a+\dfrac{b^2}{b+a}}{a-\dfrac{b^2}{b-a}}(a^6-b^6)=\dfrac{(ab+a^2+b^2)(b-a)}{(ab-a^2-b^2)(b+a)}\cdot(a^6-b^6)$

$$=\frac{b^3-a^3}{-(a^3+b^3)}\cdot(a^6-b^6)=\frac{a^3-b^3}{a^3+b^3}\cdot(a^3+b^3)(a^3-b^3)=(a^3-b^3)^2.$$

LXX.

1. $\left.\begin{array}{l}6x+21y=123\\6x+8y=84\end{array}\right\}$ subtracting, $13y=39$, $y=3$, etc.

2. $\left.\begin{array}{l}45x+72y=909\\45x+10y=475\end{array}\right\}$ subtracting, $62y=434$, $y=7$, etc.

3. $\left.\begin{array}{l}26x+34y=378\\26x+13y=273\end{array}\right\}$ subtracting, $21y=105$, $y=5$, etc.

4. $\left.\begin{array}{l}14x+9y=156\\14x+4y=116\end{array}\right\}$ subtracting, $5y=40$, $y=8$, etc.

5. $\left.\begin{array}{l}3x+45y=147\\3x+7y=71\end{array}\right\}$ subtracting, $38y=76$, $y=2$, etc.

6. $\left.\begin{array}{l}105x+133y=924\\105x+51y=678\end{array}\right\}$ subtracting, $82y=246$, $y=3$, etc.

7. $\left.\begin{array}{l}6x+4y=236\\6x+30y=1146\end{array}\right\}$ subtracting, $-26y=-910$, $y=35$, etc.

8. $\left.\begin{array}{l}156x+108y=420\\156x+87y=399\end{array}\right\}$ subtracting, $21y=21$, $y=1$, etc.

9. $\left.\begin{array}{l}504x+98y=2310\\504x+56y=2184\end{array}\right\}$ subtracting, $42y=126$, $y=3$, etc.

LXXI.

1. $\left.\begin{array}{l}6x+21y=156\\6x-10y=32\end{array}\right\}$ subtracting, $31y=124$, $y=4$, etc.

2. $\left.\begin{array}{l}105x-60y=825\\105x-91y=763\end{array}\right\}$ subtracting, $31y=62$, $y=2$, etc.

3. $\left.\begin{array}{l}x+y=96\\x-y=2\end{array}\right\}$ subtracting, $2y=94$, $y=47$, etc.

4. $\left.\begin{array}{r} 28x+63y=553 \\ 28x-68y=160 \end{array}\right\}$ subtracting, $131y=393$, $y=3$, etc.

5. $\left.\begin{array}{r} 7x+133y=679 \\ 7x-\ \ 53y=121 \end{array}\right\}$ subtracting, $186y=558$, $y=3$, etc.

6. $\left.\begin{array}{r} 87x-42y=525 \\ 87x-56y=497 \end{array}\right\}$ subtracting, $14y=28$, $y=2$, etc.

7. $\left.\begin{array}{r} 342x-426y=1284 \\ 342x-978y=\ \ 732 \end{array}\right\}$ subtracting, $552y=552$, $y=1$, etc.

8. $\left.\begin{array}{r} 516x+\ \ 24y=3192 \\ 516x-731y=\ \ 172 \end{array}\right\}$ subtracting, $755y=3020$, $y=4$, etc.

9. $\left.\begin{array}{r} 65x+117y=2444 \\ 65x-\ \ 10y=\ \ 285 \end{array}\right\}$ subtracting, $127y=2159$, $y=17$, etc.

LXXII.

1. $\left.\begin{array}{r} 12x-21y=\ \ 66 \\ 12x-28y=-4 \end{array}\right\}$ subtracting, $7y=70$, $y=10$, etc.

2. $\left.\begin{array}{r} 9x-\ \ 5y=\ \ 52 \\ 9x-24y=-24 \end{array}\right\}$ subtracting, $19y=76$, $y=4$, etc.

3. $\left.\begin{array}{r} 51x+\ \ \ 9y=\ \ 171 \\ 51x-272y=-391 \end{array}\right\}$ subtracting, $281y=562$, $y=2$, etc.

4. $\left.\begin{array}{r} 21x+49y=\ \ 546 \\ 21x-57y=-408 \end{array}\right\}$ subtracting, $106y=954$, $y=9$, etc.

5. $\left.\begin{array}{r} 35x-21y=\ \ 28 \\ 35x-60y=-50 \end{array}\right\}$ subtracting, $39y=78$, $y=2$, etc.

6. $\left.\begin{array}{r} 6x+4y=\ \ 78 \\ 6x-9y=-39 \end{array}\right\}$ subtracting, $13y=117$, $y=9$, etc.

7. $\left.\begin{array}{r}26x-65y=-273\\ 26x-8y=240\end{array}\right\}$ subtracting, $-57y=-513$, $y=9$, etc.

8. $\left.\begin{array}{r}105x-135y=-195\\ 105x-49y=63\end{array}\right\}$ subtracting, $-86y=-258$, $y=3$, etc.

9. $\left.\begin{array}{r}21y+36x=528\\ 21y-133x=21\end{array}\right\}$ subtracting, $169x=507$, $x=3$, etc.

LXXIII.

1. $\left.\begin{array}{r}6x+9y=24\\ 6x+14y=14\end{array}\right\}$ subtracting, $-5y=10$, $y=-2$, etc.

2. $\left.\begin{array}{r}15x-6y=153\\ 38x-6y=360\end{array}\right\}$ subtracting, $-23x=-207$, $x=9$, etc.

3. $\left.\begin{array}{r}6x-10y=102\\ 6x+21y=9\end{array}\right\}$ subtracting, $-31y=93$, $y=-3$.

4. $\left.\begin{array}{r}35y-15x=695\\ 35y+14x=637\end{array}\right\}$ subtracting, $-29x=58$, $x=-2$, etc.

5. $\left.\begin{array}{r}8x+18y=212\\ 8x+17y=198\end{array}\right\}$ subtracting, $y=14$, etc.

6. $\left.\begin{array}{r}18x-63y=72\\ 18x-8y=-38\end{array}\right\}$ subtracting, $-55y=110$, $y=-2$, etc.

7. $\left.\begin{array}{r}17x+12y=59\\ 57x-12y=459\end{array}\right\}$ adding, $74x=518$, $x=7$, etc.

8. $\left.\begin{array}{r}24x+9y=9\\ 12x+9y=3\end{array}\right\}$ subtracting, $12x=6$, $x=\frac{1}{2}$, etc.

9. $\left.\begin{array}{r}238x-966y=-1442\\ 238x-221y=-697\end{array}\right\}$ subtracting, $-745y=-745$, $y=1$, etc.

LXXIV.

1. $\left.\begin{array}{r}3x+2y=42\\2x+3y=48\end{array}\right\},\ \left.\begin{array}{r}6x+4y=84\\6x+9y=144\end{array}\right\},\ -5y=-60,\ y=12,$ etc.

2. $\left.\begin{array}{r}30x+y=630\\20y-x=580\end{array}\right\},\ \left.\begin{array}{r}600x+20y=12600\\20y-x=580\end{array}\right\},\ 601x=12020,\ x=20,$ etc.

3. $\left.\begin{array}{r}x+49y=1757\\y+49x=2093\end{array}\right\},\ \left.\begin{array}{r}49x+2401y=86093\\49x+y=2093\end{array}\right\},\ 2400y=84000,\ y=35,$ etc.

4. $\left.\begin{array}{r}x+y+15=30\\x-y+14=19\end{array}\right\},\ \left.\begin{array}{r}x+y=15\\x-y=5\end{array}\right\},\ 2x=20,\ x=10,$ etc.

5. $\left.\begin{array}{r}14x+5y=826\\39x-14y=-1609\end{array}\right\},\ \left.\begin{array}{r}196x+70y=11564\\195x-70y=-8045\end{array}\right\},$

$391x=3519,\ x=9,$ etc.

6. $\left.\begin{array}{r}6x+9y=150-5y\\8y-6x=9x+12\end{array}\right\},\ \left.\begin{array}{r}6x+14y=150\\8y-15x=12\end{array}\right\},\ \left.\begin{array}{r}30x+70y=750\\16y-30x=24\end{array}\right\},$

$86y=774,$ etc.

7. $\left.\begin{array}{r}7x-y+2=35\\12y-x-10=9\end{array}\right\},\ \left.\begin{array}{r}7x-y=33\\12y-x=19\end{array}\right\},\ \left.\begin{array}{r}7x-y=33\\84y-7x=133\end{array}\right\},\ 83y=166,$ etc.

8. $\left.\begin{array}{l}x+32=2y-48\\12x+12y+20y=30x-15y+2100\end{array}\right\},\ \left.\begin{array}{r}x-2y=-80\\47y-18x=2100\end{array}\right\},$

$\left.\begin{array}{r}18x-36y=-1440\\47y-18x=2100\end{array}\right\},$ adding, $11y=660,\ y=60,$ etc.

9. $\left.\begin{array}{r}15x-25y+30=4x+2y\\96-3x+6y=6x+4y\end{array}\right\},\ \left.\begin{array}{r}11x-27y=-30\\2y-9x=-96\end{array}\right\},$

$\left.\begin{array}{r}99x-243y=-270\\22y-99x=-1056\end{array}\right\},$ adding, $-221y=-1326,\ y=6,$ etc.

10. $$\left.\begin{aligned} x+2+24y&=93\\ y+5+40x&=768\end{aligned}\right\},\quad \left.\begin{aligned} x+24y&=91\\ 40x+y&=763\end{aligned}\right\},\quad \left.\begin{aligned} 40x+960y&=3640\\ 40x+y&=763\end{aligned}\right\},$$

$959y=2877$, etc.

11. $$\left.\begin{aligned} 2x-y+21x&=14y-42\\ 6y+18+5y-5x&=60x-240\end{aligned}\right\},\quad \left.\begin{aligned} 23x-15y&=-42\\ 11y-65x&=-258\end{aligned}\right\},$$

$$\left.\begin{aligned} 253x-165y&=-462\\ 165y-975x&=-3870\end{aligned}\right\} \text{adding, } -722x=-4332,\ x=6, \text{ etc.}$$

12. $$\left.\begin{aligned} 12x-24-200+20x&=15y-150\\ 16y+32&=12x+3y+39\end{aligned}\right\},\quad \left.\begin{aligned} 32x-15y&=74\\ 13y-12x&=7\end{aligned}\right\},$$

$$\left.\begin{aligned} 96x-45y&=222\\ 104y-96x&=56\end{aligned}\right\}, \text{ adding, } 59y=278,\ y=\frac{278}{59}, \text{ etc.}$$

13. $$\left.\begin{aligned} 5x-6y+39x&=52y-26\\ 10x+12y-9x+6y&=24y-24\end{aligned}\right\},\quad \left.\begin{aligned} 44x-58y&=-26\\ x-6y&=-24\end{aligned}\right\}$$

$$\left.\begin{aligned} 22x-29y&=-13\\ 22x-132y&=-528\end{aligned}\right\},\ 103y=515,\ y=5, \text{ etc.}$$

14. $$\left.\begin{aligned} 15x-9-9x+57&=24-6y+2x\\ 16x+8y-18x+14&=12y+36-4x-5y\end{aligned}\right\},\quad \left.\begin{aligned} 4x+6y&=-24\\ 2x+y&=22\end{aligned}\right\},$$

$$\left.\begin{aligned} 2x+3y&=-12\\ 2x+y&=22\end{aligned}\right\},\ 2y=-34,\ y=-17, \text{ etc.}$$

15. $$\left.\begin{aligned} 4x+5y&=40x-40y\\ 4x-2y+12y&=3\end{aligned}\right\},\quad \left.\begin{aligned} 45y-36x&=0\\ 4x+10y&=3\end{aligned}\right\},\quad \left.\begin{aligned} 5y-4x&=0\\ 4x+10y&=3\end{aligned}\right\},\ 15y=3, \text{ etc.}$$

LXXV.

1. $$\left.\begin{aligned} mpx+npy&=ep\\ mpx+mqy&=fm\end{aligned}\right\} \text{subtracting } (np-mq)y=ep-fm, \text{ etc.}$$

2. $$\left.\begin{aligned} adx+bdy&=cd\\ adx-aey&=af\end{aligned}\right\} \text{subtracting } (bd+ae)\,y=cd-af, \text{ etc.}$$

3. $$\left.\begin{array}{l} acx - bcy = cm \\ acx + aey = an \end{array}\right\}, \; y(ae + bc) = an - cm, \text{ etc.}$$

4. $$\left.\begin{array}{l} cx - dy = 0 \\ cx + cy = ce \end{array}\right\} \; (c + d)\, y = ce, \text{ etc.}$$

5. $$\left.\begin{array}{l} mm'x - m'ny = m'r \\ mm'x + mn'y = mr' \end{array}\right\}, \; (mn' + m'n)\, y = mr' - m'r, \text{ etc.}$$

6. $$\left.\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}, \; 2x = e + b, \; x = \frac{a + b}{2}, \text{ etc.}$$

7. $$\left.\begin{array}{l} adx + bdy = cd \\ adx + afy = ac^2 \end{array}\right\}, \; (af - bd)\, y = ac^2 - cd, \text{ etc.}$$

8. $$\left.\begin{array}{l} abdx + cd^2y = 2d \\ abdx - bcdy = d - b \end{array}\right\}, \; cdy\,(d + b) = d + b, \text{ etc.}$$

9. $$\left.\begin{array}{l} 3a^2 + ax = b^2 + by \\ ax + 2by = d \end{array}\right\}, \; \left.\begin{array}{l} ax - by = b^2 - 3a^2 \\ ax + 2by = d \end{array}\right\}, \; 3by = 3a^2 - b^2 + d, \text{ etc.}$$

10. $$\left.\begin{array}{l} bcx - cy = -2b \\ b^3cy + ac^3 - ab^3 = 2b^4 + bc^4x \end{array}\right\}, \; \left.\begin{array}{l} bc^4x - c^4y = -2bc^3 \\ b^3cy - bc^4x = 2b^4 - ac^3 + ab^3 \end{array}\right\},$$

$$b^3cy - c^4y = 2b^4 - ac^3 + ab^3 - 2bc^3,$$

$$y = \frac{2b^4 - ac^3 + ab^3 - 2bc^3}{c\,(b^3 - c^3)} = \frac{2b + a}{c}, \text{ etc.}$$

11. $$\left.\begin{array}{l} bx + bc - b^2 + cx + c^2 - bc + ay + a^2 = 2a^2 \\ a^3y = (b + c)^2 (b - c)\, x \end{array}\right\},$$

$$\left.\begin{array}{l} (b + c)\, x + ay = a^2 + b^2 - c^2 \\ (b + c)\,(b^2 - c^2)\, x - a^3y = 0 \end{array}\right\}, \; \left.\begin{array}{l} a^2(b + c)x + a^3y = a^2(a^2 + b^2 - c^2) \\ (b + c)\,(b^2 - c^2)\, x - a^3y = 0 \end{array}\right\},$$

$$(b + c)\,(a^2 + b^2 - c^2)\, x = a^2\,(a^2 + b^2 - c^2)\;;\; x = \frac{a^2}{b + c}, \text{ etc.}$$

12. $$\left.\begin{array}{l} 3x + 5y = \dfrac{8b^2m - 2bm^2}{b^2 - m^2} \\ (b^2 - m^2)x + (b + c + m)\, my = b^2m + 2bm^2 + \dfrac{bcm^2}{b + m} \end{array}\right\}$$

$$\left.\begin{aligned}&3\,(b^2-m^2)\,x+5\,(b^2-m^2)\,y=8b^2m-2bm^2\\&3(b^2-m^2)x+3(bm+cm+m^2)=3b^2m+6bm^2+\frac{3bcm^2}{b+m}\end{aligned}\right\}$$

$$\begin{aligned}(5b^2-3bm-3cm-8m^2)\,y&=5b^2m-8bm^2-\frac{3bcm^2}{b+m}\\&=\frac{5b^3m+5b^2m^2-8b^2m^2-8bm^3-3bcm^2}{b+m}\\&=\frac{bm\,(5b^2-3bm-3cm-8m^2)}{b+m},\text{ etc.}\end{aligned}$$

LXXVI.

1. $$\left.\begin{aligned}&\frac{4}{x}+\frac{8}{y}=40\\&\frac{4}{x}+\frac{3}{y}=20\end{aligned}\right\}\text{ subtracting, }\frac{5}{y}=20\,;\ y=\frac{1}{4},\text{ etc.}$$

2. $$\left.\begin{aligned}&\frac{3}{x}+\frac{6}{y}=3a\\&\frac{3}{x}+\frac{4}{y}=b\end{aligned}\right\}\text{ subtracting, }\frac{2}{y}=3a-b\,;\ y=\frac{2}{3a-b},\text{ etc.}$$

3. $$\left.\begin{aligned}&\frac{ab}{x}+\frac{b^2}{y}=bc\\&\frac{ab}{x}+\frac{a^2}{y}=ad\end{aligned}\right\}\text{ subtracting, }\frac{b^2-a^2}{y}=bc-ad\,;\ y=\frac{b^2-a^2}{bc-ad},\text{ etc.}$$

4. $$\left.\begin{aligned}&\frac{a}{x}+\frac{b}{y}=m\\&\frac{a}{x}-\frac{b}{y}=n\end{aligned}\right\}\text{ subtracting, }\frac{2b}{y}=m-n\,;\ y=\frac{2b}{m-n},\text{ etc.}$$

5. $$\left.\begin{aligned}&\frac{56}{x}+\frac{40}{y}=152\\&\frac{56}{x}-\frac{21}{y}=\ \ 49\end{aligned}\right\}\text{ subtracting, }\frac{61}{y}=103\,;\ y=\frac{61}{103},\text{ etc.}$$

6. Multiply the second equation by 4, and we get

$$\left.\begin{array}{r}\dfrac{5}{3x}+\dfrac{2}{5y}=\ \ 7\\[2ex] \dfrac{14}{3x}-\dfrac{2}{5y}=12\end{array}\right\}\text{ adding, }\frac{19}{3x}=19\text{ ; }x=\frac{1}{3},\text{ etc.}$$

7. $$\left.\begin{array}{r}\dfrac{4}{ax}+\dfrac{6}{by}=10\\[2ex] \dfrac{15}{ax}-\dfrac{6}{by}=\ \ 9\end{array}\right\}\text{ adding, }\frac{19}{ax}=19\text{ ; }x=\frac{1}{a},\text{ etc.}$$

8. Multiply the first equation by $\frac{n^2}{m}$, and we get

$$\left.\begin{array}{l}\dfrac{n}{x}+\dfrac{n^3}{m^2y}=\dfrac{mn^2+n^3}{m}\\[2ex] \dfrac{n}{x}+\dfrac{m}{y}\ \ =m^2+n^2\end{array}\right\}\text{ subtracting, }\frac{n^3-m^3}{m^2y}=\frac{n^3-m^3}{m}\text{ ; }y=\frac{1}{m},\text{ etc.}$$

LXXVII.

1. From the first and second equations we get

$$\left.\begin{array}{r}5x+7y-2z=13\\ 16x+6y+2z=34\end{array}\right\}\text{ adding, }21x+13y=47\ (1)$$

From the first and third equations we get

$$\left.\begin{array}{r}25x+35y-10z=65\\ x-\ \ 4y+10z=23\end{array}\right\}\text{ adding, }26x+31y=88\ (2)$$

Then from (1) and (2) we find $x=1$, $y=2$; and hence $z=3$.

2. From the first and second equations we get

$$\left.\begin{array}{r}5x+3y-6z=\ \ 4\\ 9x-3y+6z=24\end{array}\right\}\text{ adding, }14x=28\text{ ; }x=2$$

From the second and third we get $2x+y=6$; $\therefore y=2$, etc.

3. From the first and second equations we get

$$\left.\begin{array}{r}15x-9y+6z=63\\ 16x-2y-6z=\ \ 6\end{array}\right\}\text{ adding, }31x-11y=69\ (1)$$

From the first and third, subtracting, $3x-6y=-18$ (2)

Then from (1) and (2) we find $x=4$, $y=5$; and $\therefore z=8$.

4. From the first and second equations we get

$$\left.\begin{array}{r}4x-5y+2z=6\\4x+6y-2z=40\end{array}\right\}\text{ adding, } 8x+y=46 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{array}{r}6x+9y-3z=60\\7x-4y+3z=35\end{array}\right\}\text{ adding, } 13x+5y=95 \text{ (2)}$$

Then from (1) and (2) we find $x=5$, $y=6$; and $\therefore z=8$.

5. From the second and first equations we get

$$\left.\begin{array}{r}5x+4y+3z=22\\3x+3y+3z=18\end{array}\right\}\text{ subtracting, } 2x+y=4 \text{ (1)}$$

From the third and second equations we get

$$\left.\begin{array}{r}15x+10y+6z=53\\10x+8y+6z=44\end{array}\right\}\text{ subtracting, } 5x+2y=9 \text{ (2)}$$

Then from (1) and (2) we find $x=1$, $y=2$; and $\therefore z=3$.

6. From the first and second equations we get

$$\left.\begin{array}{r}8x+4y-3z=6\\3x+9y-3z=21\end{array}\right\}\text{ subtracting, } 5x-5y=-15 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{array}{r}4x+12y-4z=28\\4x-5y+4z=8\end{array}\right\}\text{ adding, } 8x+7y=36 \text{ (2)}$$

Then from (1) and (2) we find $x=1$, $y=4$; and $\therefore z=6$.

7. From the second and first equations we get

$$\left.\begin{array}{r}8x+4y+2z=50\\2x+2y+2z=60\end{array}\right\}\text{ subtracting, } 6x+2y=-10 \text{ (1)}$$

From the third and first equations we get

$$\left.\begin{array}{r} 27x+9y+3z=64 \\ 3x+3y+3z=90 \end{array}\right\} \text{subtracting, } 24x+6y=-26 \ (2)$$

Then from (1) and (2) we find $x=\frac{2}{3}$, $y=-7$; and $\therefore z=36\frac{1}{3}$.

8. From the first and second equations we get

$$\left.\begin{array}{r} 4x-3y+\quad z=\ \ 9 \\ 27x+3y-15z=48 \end{array}\right\} \text{adding, } 31x-14z=57 \ (1)$$

From the second and third equations we get

$$\left.\begin{array}{r} 36x+4y-20z=64 \\ x-4y+\ \ 3z=\ \ 2 \end{array}\right\} \text{adding, } 37x-17z=66 \ (2)$$

Then from (1) and (2) we find $x=5$, $z=7$; and $\therefore y=6$.

9. From the first and second equations we get

$$\left.\begin{array}{r} 24x+10y-\ \ 8z=\ \ 58 \\ 65x-10y+25z=290 \end{array}\right\} \text{adding, } 89x+17z=348 \ (1)$$

From the first and third equations we get

$$\left.\begin{array}{r} 12x+5y-4z=29 \\ 85x-5y-5z=75 \end{array}\right\} \text{adding, } 97x-9z=104 \ (2)$$

Then from (1) and (2) we find $x=2$, $z=10$; and $\therefore y=9$.

10. Subtracting the second equation from the first, we get $2y=20$; $y=10$.

Adding the third equation to the second, we get $2z=10$; $z=5$.

LXXVIII.

1. Let x and y be the numbers; then $x+y=28$; $x-y=4$, etc.
2. Let x and y be the numbers; then $x+y=256$; $x-y=10$, etc.
3. Let x and y be the numbers; then $x+y=13{\cdot}5$; $x-y=1$, etc.

4. Let x and y be the numbers ; then $7x+5y=332$;
$51x-51y=408$, etc.

5. Let x be the age of the father, and y the age of the son.
Then $x-7=4(y-7)$ and $x+7=2(y+7)$, etc.

6. Let x, y, z be the numbers ; then $x+y=70, x+z=80, y+z=90$, etc.

7. Let x and y be the sums contributed by A and B ;
then $400-(x+y)$ is the sum contributed by C.
Then $y=2x+20$, and $400-(x+y)=x+y$, etc.

8. Let x be A's money, and y B's money in shillings.
Then $3(x-10)=y+10$, and $x+10=2(y-10)$, etc.

9. Let x and y be A's and B's shares ; then $760-(x+y)$ is C's share.
Then $x+y=760-x-y+240$; $y+760-x-y=x+360$, etc.

10. Let x and y be the numbers ; then $\frac{x+y}{2}=24$; $\frac{x-y}{2}=17$, etc.

11. Let x be the greater number ; and y the less ; then
$\frac{x}{y}=4+\frac{3}{y}$; $\frac{x+y+38}{x}=2+\frac{2}{x}$, etc.

12. Let x be the first part, y the second, and $\therefore$ $144-(x+y)$ the third.
Then $\frac{x}{y}=3+\frac{2}{y}$; $\frac{144-x-y}{x+y}=2+\frac{6}{x+y}$, etc.

13. Let x be A's money and y B's money in pounds.
Then $x+\frac{y}{2}=120$; $\frac{2x}{3}+y=120$, etc.

14. Let x be the age of the father, y the age of the son.
Then $x-12=y+12$; $x+12=3(y-12)$, etc.

15. Let x be the greater number, and y the less.
Then $3x = 2x + 10$, and $2x + 3y = 24$, etc.

16. Let x be the age of the father, and y the age of the son.
Then $x + y = \frac{1}{2}(x + y + 50)$; $x - y = \frac{1}{3}(x + y + 40)$, etc.

17. Let x be the greater number, and y the less.
Then $\frac{y}{x} = \cdot 21 + \frac{\cdot 0057}{x}$; $\frac{x}{y} = 4 + \frac{\cdot 742}{y}$, etc.

18. Let x be the cost of a barrel of beer, y of a barrel of porter in pounds.
Then $6x + 10y = 51$; $3x + 7y = 32\frac{1}{10}$. Hence we find $x = 3$, and hence 10 barrels of beer can be bought for £30.

19. Let x be the cost of 1 lb. of tea, y of 1 lb. of coffee in pence.
Then $7x + 5y = 352$, and $4x + 9y = 324$, etc.

20. Let x be the cost of a horse, y of a cow in pounds.
Then $12x + 14y = 380$, and $5x + 3y = 130$, etc.

21. Let x be the cost of a yard of silk, y of a yard of cloth in pence.
Then $8x + 19y = 4370$, and $20x + 16y = 6200$, etc.

22. In one day 10 men and 6 women earn £3, 3s.
In one day 4 men and 8 women earn £2, 2s.
Let x be the daily earnings of a man, y of a woman in shillings.
Then $10x + 6y = 63$, and $4x + 8y = 42$, etc.

23. Let x be the number at £37; and y the number at £45.
Then $37x + 45y = 4220$; and $x + y = 100$, etc.

24. Let $10x + y$ be the number.
Then $x + y = 8$; and $10x + y + 36 = 10y + x$, etc.

25. Let $10x + y$ be the number.
Then $x + y = 10$; and $10x + y + 54 = 10y + x$, etc.

26. Let $10x + y$ be the number.
Then $x + y = 9$; and $10x + y + 9 = 10y + x$, etc.

27. Let $10x+y$ be the number.
Then $x+y=6$; and $\dfrac{10x+y}{x+y}=4$, etc.

28. Let $10x+y$ be the number.
Then $x+y=9$; and $\dfrac{10x+y}{x+y}=5$, etc.

29. Let $10x+y$ be the number.
Then $\dfrac{10x+y}{x+y}=7$; and $\dfrac{10y+x-12}{x-y}=9$, etc.

30. Let $10x+y$ be the number.
Then $\dfrac{10x+y}{x+y}=6+\dfrac{3}{x+y}$; and $\dfrac{10y+x}{x+y}=4+\dfrac{9}{x+y}$
or, $10x+y=6x+6y+3$; and $10y+x=4x+4y+9$, etc.

31. Let $10x+y$ be the number.
Then $\dfrac{10x+y}{x+y-2}=5+\dfrac{1}{x+y-2}$; $\dfrac{10y+x}{x+y+2}=5+\dfrac{8}{x+y+2}$, etc.

32. Let $10x+y$ and $10y+x$ be the numbers.
Then $10x+y+9=10y+x$, and $10x+y+10y+x=33$, etc.

33. Let $100x+10x+x$ be the number.
Then $100x+10x+x=37x^2$; $\therefore x=3$, etc.

34. Let $100x+10y+z$ be the number.
Then $y=2z$; $x+z=9$; $x+y+z=17$.
Hence $x=9-z$; and $\therefore 9-z+2z+z=17$; $z=4$, etc.

35. Let $100x+10y+z$ be the number.
Then $x+y+2=21$; $x+y=z+3$;
$100x+10y+z+198=100z+10y+x$, etc.

36. Let $\dfrac{x}{y}$ be the fraction. Then $\dfrac{x+7}{y}=2$, and $\dfrac{x}{y-1}=1$, etc.

37. Let $\dfrac{x}{y}$ be the fraction. Then $\dfrac{x+1}{y}=\dfrac{1}{3}$, and $\dfrac{x}{y+1}=\dfrac{1}{4}$, etc.

38. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+1}=\frac{1}{3}$, etc.

39. Let $\frac{x}{y}$ be the fraction. Then $x+1=y$, and $x=\frac{1}{2}(y+1)$, etc.

40. Let $\frac{x}{y}$ be the fraction. Then $\frac{x-3}{y-3}=\frac{1}{4}$, and $\frac{x+5}{y+5}=\frac{1}{2}$, etc.

41. Let $\frac{x}{y}$ be the fraction. Then $\frac{x}{y+4}=\frac{7}{9}$, and $\frac{x-15}{y}=\frac{20}{41}$, etc.

42. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+17}=\frac{1}{3}$, etc.

43. Let x and y be the sums invested.

Then $\frac{x\times 5}{100}=$ income on first investment.

and $\frac{y\times 4}{100}=$ income on second investment.

$\therefore \frac{x\times 5}{100}=\frac{y\times 4}{100}+10$; and $x+y=2000$, etc.

44. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 10}{100\times 12}=5250$

and $x+\frac{x\times y\times 18}{100\times 12}=5450.$

$$\text{Hence }\left.\begin{aligned}120x+\ \ xy&=\ \ 630000\\ \text{and }\ 200x+3xy&=1090000\end{aligned}\right\},\quad \left.\begin{aligned}360x+3xy&=1890000\\ 200x+3xy&=1090000\end{aligned}\right\},$$

$160x=800000$, etc.

45. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 6}{100}=5200.$

and $x+\frac{x\times y\times 10}{100}=6000$

$$\text{Hence }\left.\begin{aligned}50x+3xy&=260000\\ 10x+\ \ xy&=\ \ 60000\end{aligned}\right\},\quad \left.\begin{aligned}50x+3xy&=260000\\ 30x+3xy&=180000\end{aligned}\right\},$$

$20x=80000$, etc.

46. Let x be the number of quarts of the first, y of the second.

Then $\left.\begin{array}{r} x+\ y\ =50 \\ 36x+20y=30\times 50 \end{array}\right\}, \left.\begin{array}{r} 36x+36y=1800 \\ 36x+20y=1500 \end{array}\right\}, 16y=300$, etc.

47. Let x be the number of lbs. of the cheaper, y of the dearer.

Then $x+y=30$

and $20(x+y)=14x+18y+10x$, etc.

48. Let x be the rate of the rowing in miles an hour in still water, y the rate of the stream in miles an hour.

Then $x+y=12$, $x-y=6$, etc.

49. Let x be the distance in miles, y the rate in miles an hour of pulling.

Then $x=1\frac{2}{3}(y+4)$

and $x-3=4\frac{1}{4}(y-4)$, etc.

50. Let x be the number of leaps the hare takes, and y the length of each in feet.

Then $\frac{5x}{6}$ is the number of leaps the dog takes, and $\frac{9y}{7}$ the length of each in feet.

Then $50y+xy=\frac{5x}{6}\times\frac{9y}{7}$

Divide by y; then $50+x=\frac{15x}{14}$; $x=700$.

51. Let x be the number of leaps the dog takes, and y the length of each in feet.

Then $\frac{4x}{3}$ is the number of leaps the hare takes, and $\frac{2y}{3}$ is the length of each in feet.

Then $50y+\frac{4x}{3}\times\frac{2y}{3}=xy$.

Divide by y; then $50+\frac{8x}{9}=x$; $x=450$, etc.

52. Let x be the number of apples, and y the number of pears.

Then $\frac{x}{4}+\frac{y}{5}=30$; and $\frac{x}{8}+\frac{y}{15}=13$, etc.

53. Let x be the number of men, y the reckoning of each in shillings.

$$\left.\begin{array}{l}\text{Then } (x+3)(y-1)=xy \\ \text{and } \quad (x-2)(y+1)=xy\end{array}\right\}, \quad \left.\begin{array}{l}xy+3y-x-3=xy \\ xy-2y+x-2=xy\end{array}\right\},$$

$$\left.\begin{array}{r}3y-x=3 \\ x-2y=2\end{array}\right\} \text{ adding, } y=5, \text{ and hence } x=12.$$

54. Let x be the number that voted for A and C,

y „ „ for A and B,

z „ „ for A only.

$$\text{Then } \left.\begin{array}{r}x+\ y+\ z=1056 \\ x+85+\ 98=933 \\ y+85+744=987\end{array}\right\}, \text{ hence } x=750,\ y=158,\ z=148.$$

55. Let x be the distance in miles; y the rate in miles an hour.

$$\left.\begin{array}{l}\text{Then } \dfrac{x}{y+\frac{1}{2}}=\dfrac{x}{y}-1\tfrac{1}{2} \\ \text{and } \quad \dfrac{x}{y-\frac{1}{2}}=\dfrac{x}{y}+2\tfrac{1}{2}\end{array}\right\}, \text{ or, } \left.\begin{array}{l}\dfrac{2x}{2y+1}=\dfrac{2x-3y}{2y} \\ \dfrac{2x}{2y-1}=\dfrac{2x+5y}{2y}\end{array}\right\}.$$

$$\left.\begin{array}{l}4xy=4xy+2x-\ 6y^2-3y \\ 4xy=4xy-2x+10y^2-5y\end{array}\right\}, \quad \left.\begin{array}{l}2x=\ 6y^2+3y \\ 2x=10y^2-5y\end{array}\right\}$$

hence $6y^2+3y=10y^2-5y$; $8y=4y^2$; $8=4y$; $y=2$, etc.

56. Suppose the first crew pulls x strokes of y yards each in a minute.

Then the second crew pulls $\frac{8x}{9}$ strokes of $\frac{90y}{79}$ yards each in a minute.

Hence in one minute the first crew pulls over xy yards,

and in one minute the second crew pulls over $\frac{80xy}{79}$ yards;

∴ the second is the faster crew.

Again, the second crew gains $\frac{xy}{79}$ yards in a minute;

∴ it gains $\frac{xy}{79}\div\frac{8x}{9}$ yards in a stroke, or, $\frac{9y}{79\times 8}$ yards.

Now it has to gain $\frac{4\times 90y}{79}$ yards;

$\therefore$ it must take $\frac{4\times 90y}{79}\div\frac{9y}{79\times 8}$ strokes, or, 320 strokes.

57, Let x be the rate of the sculler; y the rate of the barges.

Then $\frac{b}{x}=$ time he takes to meet the first barge

$\frac{a-b}{y}=$ time the first barge takes to meet him.

Hence $\frac{b}{x}=\frac{a-b}{y}$, or, $\frac{b}{a-b}=\frac{x}{y}$. (1)

Also $\frac{b'}{x}=$ time he takes to overtake the second barge

$\frac{b'-a}{y}=$ time the second barge is in motion before he overtakes it;

$\therefore\ \frac{b'}{x}=\frac{b'-a}{y}$, or, $\frac{b'}{b'-a}=\frac{x}{y}$ (2)

From (1) and (2) $\frac{b}{a-b}=\frac{b'}{b'-a}$, or, $bb'-ab=ab'-bb'$,

or $2bb'=ab+ab'$, or, $\frac{2bb'}{abb'}=\frac{ab}{abb'}+\frac{ab'}{abb'}$, or, $\frac{2}{a}=\frac{1}{b'}+\frac{1}{b}$.

58. Let x be the number of feet passed over by the longer train in a second,

y the number of feet passed over by the shorter train in a second.

Then $\frac{3x}{2}+\frac{3y}{2}=176$, and $6x-6y=176$.

Hence $x=\frac{220}{3}$ and $y=44$.

Hence the longer train goes in miles per hour $\frac{220\times 60\times 60}{3\times 3\times 1760}$, or, 50.

and the shorter train goes in miles per hour $\frac{44\times 60\times 60}{3\times 1760}$, or, 30.

59. Let x and y be the circumferences of the wheels in yards.

$$\text{Then } \frac{120}{x}=\frac{120}{y}+6, \text{ and } \frac{120}{x+\frac{x}{4}}=\frac{120}{y+\frac{y}{5}}+4;$$

$$\text{hence } \frac{20}{x}-\frac{20}{y}=1, \text{ and } \frac{96}{x}-\frac{100}{y}=4; \therefore x=4, y=5.$$

60. Let x be the number of hours he takes to go.

Then $10-x$ is the number of hours he takes to return,

and $x=\frac{2}{3}(10-x)$; whence $x=4$, and $10-x=6$.

Again let y be the number of miles per hour he can row in still water, and z the number of miles per hour the stream flows.

Then $y+z=5$, and $y-z=\frac{20}{6}$; whence $z=\frac{5}{6}$.

61. Let the digits commencing with the left be $1, a, b, c, d, e$.

Then the number is $100000+10000a+1000b+100c+10d+e$.

When the 1 is removed to the unit's place
the new number is $100000a+10000b+1000c+100d+10e+1$.

Then $100000a+10000b+1000c+100d+10e+1$

$=300000+30000a+3000b+300c+30d+3e$

$\therefore$ $70000a+7000b+700c+70d+7e=299999$
or, $10000a+1000b+100c+10d+e=42857$
$\therefore$ $100000+10000a+1000b+100c+10d+e=142857$
that is, the required number is 142857.

LXXX.

1.

$$\begin{array}{r|l} & 4a^2+12ab+9b^2\,(2a+3b \\ & 4a^2 \\ \hline 4a+3b & \quad 12ab+9b^2 \\ & \quad 12ab+9b^2 \\ \hline \end{array}$$

2.

$$\begin{array}{r|l} & 16k^{10}-24k^5l^3+9l^6\,(4k^5-3l^3 \\ & 16k^{10} \\ \hline 8k^5-3l^3 & \quad -24k^5l^3+9l^6 \\ & \quad -24k^5l^3+9l^6 \\ \hline \end{array}$$

3.
$$\begin{array}{r|l}
 & a^2b^2+162ab+6561\ (ab+81 \\
 & a^2b^2 \\ \hline
2ab+81 & 162ab+6561 \\
 & 162ab+6561 \\ \hline
\end{array}$$

4.
$$\begin{array}{r|l}
 & y^6-38y^3+361\ (y^3-19 \\
 & y^6 \\ \hline
2y^3-19 & -38y^3+361 \\
 & -38y^3+361 \\ \hline
\end{array}$$

5.
$$\begin{array}{r|l}
 & 9a^2b^2c^2-102abc+289\ (3abc-17 \\
 & 9a^2b^2c^2 \\ \hline
6abc-17 & -102abc+289 \\
 & -102abc+289 \\ \hline
\end{array}$$

6.
$$\begin{array}{r|l}
 & x^4-6x^3+19x^2-30x+25\ (x^2-3x+5 \\
 & x^4 \\ \hline
2x^2-3x & -6x^3+19x^2 \\
 & -6x^3+\ \ 9x^2 \\ \hline
2x^2-6x+5 & 10x^2-30x+25 \\
 & 10x^2-30x+25 \\ \hline
\end{array}$$

7.
$$\begin{array}{r|l}
 & 9x^4+12x^3+10x^2+4x+1\ (3x^2+2x+1 \\
 & 9x^4 \\ \hline
6x^2+2x & 12x^3+10x^2 \\
 & 12x^3+\ \ 4x^2 \\ \hline
6x^2+4x+1 & 6x^2+4x+1 \\
 & 6x^2+4x+1 \\ \hline
\end{array}$$

8.
$$\begin{array}{r|l}
 & 4r^4-12r^3+13r^2-6r+1\ (2r^2-3r+1 \\
 & 4r^4 \\ \hline
4r^2-3r & -12r^3+13r^2 \\
 & -12r^3+\ \ 9r^2 \\ \hline
4r^2-6r+1 & 4r^2-6r+1 \\
 & 4r^2-6r+1 \\ \hline
\end{array}$$

9.

$$\begin{array}{ll}
 & 4n^4+4n^3-7n^2-4n+4\ (2n^2+n-2 \\
 & 4n^4 \\
\hline
4n^2+n & 4n^3-7n^2 \\
 & 4n^3+\ n^2 \\
\hline
4n^2+2n-2 & -8n^2-4n+4 \\
 & -8n^2-4n+4 \\
\hline
\end{array}$$

10.

$$\begin{array}{ll}
 & 1\ -6x+13x^2+12x^3+4x4\ (1-3x+2x^2 \\
 & 1 \\
\hline
2-3x & -6x+13x^2 \\
 & -6x+\ 9x^2 \\
\hline
2-6x+2x^2 & 4x^2+12x^3+4x^4 \\
 & 4x^4-12x^3+4x^4 \\
\hline
\end{array}$$

11.

$$\begin{array}{ll}
 & x^6-4x^5+10x^4-12x^3+9x^2\ (x^3-2x^2+3x \\
 & x6 \\
\hline
2x^3-2x^2 & -4x^5+10x^4 \\
 & -4x^5+\ 4x^4 \\
\hline
2x^3-4x^2+3x & 6x^4-12x^3+9x^2 \\
 & 6x^4-12x^3+9x^2 \\
\hline
\end{array}$$

12.

$$\begin{array}{ll}
 & 4y^4-12y^3z+25y^2z^2-24yz^3+16z^4\ (2y^2-3yz+4 \\
 & 4y^4 \\
\hline
4y^2-3yz & -12y^3z+25y^2z^2 \\
 & -12y^3z+9y^2z^2 \\
\hline
4y^2-6yz+4z^2 & 16y^2z^2-24yz^3+16z^4 \\
 & 16y^2z^2-24yz^3+16z^4 \\
\hline
\end{array}$$

13.

$$\begin{array}{ll}
 & a^2+4ab+4b^2+9c^2+6ac+12bc\,(a+2b+3c \\
 & a^2 \\ \cline{2-2}
2a+2b & 4ab+4b^2 \\
 & 4ab+4b^2 \\ \cline{2-2}
2a+4b+3c & 6ac+12bc+9c^2 \\
 & 6ac+12bc+9c^2 \\ \cline{2-2}
\end{array}$$

14.

$$\begin{array}{ll}
 & a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6\,(a^3+a^2b+ab^2+b^3 \\
 & a^6 \\ \cline{2-2}
2a^3+a^2b & 2a^5b+3a^4b^2 \\
 & 2a^5b+a^4b^2 \\ \cline{2-2}
2a^3+2a^2b+ab^2 & 2a^4b^2+4a^3b^3+3a^2b^4 \\
 & 2a^4b^2+2a^3b^3+a^2b^4 \\ \cline{2-2}
2a^3+2a^2b+2ab^2+b^3 & 2a^3b^3+2a^2b^4+2ab^5+b^6 \\
 & 2a^3b^3+2a^2b^4+2ab^5+b^6 \\ \cline{2-2}
\end{array}$$

15.

$$\begin{array}{ll}
 & x^6-4x^5+6x^3+8x^2+4x+1\,(x^3-2x^2-2x-1 \\
 & x^6 \\ \cline{2-2}
2x^3-2x^2 & -4x^5+6x^3 \\
 & -4x^5+4x^4 \\ \cline{2-2}
2x^3-4x^2-2x & -4x^4+6x^3+8x^2 \\
 & -4x^4+8x^3+4x^2 \\ \cline{2-2}
2x^3-4x^2-4x-1 & -2x^3+4x^2+4x+1 \\
 & -2x^3+4x^2+4x+1 \\ \cline{2-2}
\end{array}$$

16.

$$\begin{array}{ll}
 & 4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4\,(2x^2+2ax+4b \\
 & 4x^4 \\ \cline{2-2}
4x^2+2ax & 8ax^3+4a^2x^2 \\
 & 8ax^3+4a^2x^2 \\ \cline{2-2}
4x^2+4ax+4b^2 & 16b^2x^2+16ab^2x+16b^4 \\
 & 16b^2x^2+16ab^2x+16b^4 \\ \cline{2-2}
\end{array}$$

17.

$$\begin{array}{l|l}
 & 9-24x+58x^2-116x^3+129x^4-140x^5+100x^6\,(3-4x+7x^2-10x^3 \\
 & 9 \\ \hline
6-4x & -24x+58x^2 \\
 & -24x+16x^2 \\ \hline
6-8x+7x^2 & \qquad 42x^2-116x^3+129x^4 \\
 & \qquad 42x^2-56x^3+49x^4 \\ \hline
6-8x+14x^2-10x^3 & \qquad\qquad -60x^3+80x^4-140x^5+100x^6 \\
 & \qquad\qquad -60x^3+80x^4-140x^5+100x^6 \\ \hline
\end{array}$$

18.

$$\begin{array}{l|l}
 & 16a^4-40a^3b+25a^2b^2-80ab^2x+64b^2x^2+64a^2bx\,(4a^2-5ab+8bx \\
 & 16a^4 \\ \hline
8a^2-5ab & -40a^3b+25a^2b^2 \\
 & -40a^3b+25a^2b^2 \\ \hline
8a^2-10ab+8bx & \qquad\qquad 64a^2bx-80ab^2x+64b^2x^2 \\
 & \qquad\qquad 64a^2bx-80ab^2x+64b^2x^2 \\ \hline
\end{array}$$

19.

$$\begin{array}{l|l}
 & 9a^4-24a^3p^3-30a^2t+16a^2p^6+40ap^3t+25t^2\,(3a^2-4ap^3-5t \\
 & 9a^4 \\ \hline
6a^2-4ap^3 & -24a^3p^3+16a^2p^6 \\
 & -24a^3p^3+16a^2p^6 \\ \hline
6a^2-8ap^3-5t & \qquad -30a^2t+40ap^3t+25t^2 \\
 & \qquad -30a^2t+40ap^3t+25t^2 \\ \hline
\end{array}$$

20.

$$\begin{array}{l|l}
 & 4y^4x^2-12y^3x^3+17y^2x^4-12yx^5+4x^6\,(2y^2x-3yx^2+2x^3 \\
 & 4y^4x^2 \\ \hline
4y^2x-3yx^2 & -12y^3x^3+17y^2x^4 \\
 & -12y^3x^3+9y^2x^4 \\ \hline
4y^2x-6yx^2+2x^3 & \qquad\qquad 8y^2x^4-12yx^5+4x^6 \\
 & \qquad\qquad 8y^2x^4-12yx^5+4x^6 \\ \hline
\end{array}$$

21.

$$\begin{array}{l|l}
 & 25x^4y^2 - 30x^3y^3 + 29x^2y^4 - 12xy^5 + 4y^6\,(5x^2y - 3xy^2 + 2y^3 \\
 & 25x^4y^2 \\ \hline
10x^2y - 3xy^2 & -30x^3y^3 + 29x^2y^4 \\
 & -30x^3y^3 + 9x^2y^2 \\ \hline
10x^2y - 6xy^2 + 2y^3 & 20x^2y^4 - 12xy^5 + 4y^6 \\
 & 20x^2y^4 - 12xy^5 + 4y^6 \\ \hline
\end{array}$$

22.

$$\begin{array}{l|l}
 & 16x^4 - 24x^3y + 25x^2y^2 - 12xy^3 + 4y^4\,(4x^2 - 3xy + 2y^2 \\
 & 16x^4 \\ \hline
8x^2 - 3xy & -24x^3y + 25x^2y^2 \\
 & -24x^3y + 9x^2y^2 \\ \hline
8x^2 - 6xy + 2y^2 & 16x^2y^2 - 12xy^3 + 4y^4 \\
 & 16x^2y^2 - 12xy^3 + 4y^4 \\ \hline
\end{array}$$

23.

$$\begin{array}{l|l}
 & 9a^2 - 12ab + 24ac - 16bc + 4b^2 + 16c^2\,(3a - 2b + 4c \\
 & 9a^2 \\ \hline
6a - 2b & -12ab + 4b^2 \\
 & -12ab + 4b^2 \\ \hline
6a - 4b + 4c & 24ac - 16bc + 16c^2 \\
 & 24ac - 16bc + 16c^2 \\ \hline
\end{array}$$

24.

$$\begin{array}{l|l}
 & x^4 + 19x^2 + 25 - 6x^3 - 30x\,(x^2 - 3x + 5 \\
 & x^4 \\ \hline
2x^2 - 3x & -6x^3 + 19x^2 \\
 & -6x^3 + 9x^2 \\ \hline
2x^2 - 6x + 5 & 10x^2 - 30x + 25 \\
 & 10x^2 - 30x + 25 \\ \hline
\end{array}$$

25.

$$\begin{array}{rl}
 & 25x^2 - 20xy + 4y^2 + 9z^2 - 12yz + 30xz\ (5x - 2y + 3z \\
 & 25x^2 \\ \hline
10x - 2y & -20xy + 4y^2 \\
 & -20xy + 4y^2 \\ \hline
10x - 4y + 3z & 30xz - 12yz + 9z^2 \\
 & 30xz - 12yz + 9z^2 \\ \hline
\end{array}$$

26.

$$\begin{array}{rl}
 & 4x^4 - 4x^2y + 4x^2y^2 + y^4 - 2y^3 + y^2\ (2x^2 - y + y^2 \\
 & 4x^4 \\ \hline
4x^2 - y & -4x^2y + y^2 \\
 & -4x^2y + y^2 \\ \hline
4x^2 - 2y + y^2 & 4x^2y^2 - 2y^3 + y^4 \\
 & 4x^2y^2 - 2y^3 + y^4 \\ \hline
\end{array}$$

LXXXI.

1.

$$\begin{array}{rl}
 & 4a^6 + \frac{a^2b^4}{16} - a^4b^2\ \left(2a^3 - \frac{ab^2}{4}\right. \\
 & 4a^6 \\ \hline
4a^3 - \frac{ab^2}{4} & -a^4b^2 + \frac{a^2b^4}{16} \\
 & -a^4b^2 + \frac{a^2b^4}{16} \\ \hline
\end{array}$$

2.

$$\begin{array}{rl}
 & \frac{9}{a^2} - 2 + \frac{a^2}{9}\left(\frac{3}{a} - \frac{a}{3}\right. \\
 & \frac{9}{a^2} \\ \hline
\frac{6}{a} - \frac{a}{3} & -2 + \frac{a^2}{9} \\
 & -2 + \frac{a^2}{9} \\ \hline
\end{array}$$

3.
$$\begin{array}{l|l}
 & a^4 - 2 + \frac{1}{a^4}\left(a^2 - \frac{1}{a^2}\right. \\
 & a^4 \\ \hline
2a^2 - \frac{1}{a^2} & -2 + \frac{1}{a^4} \\
 & -2 + \frac{1}{a^4} \\ \hline
\end{array}$$

4.
$$\begin{array}{l|l}
 & \frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}\left(\frac{a}{b} + \frac{b}{a}\right. \\
 & \frac{a^2}{b^2} \\ \hline
\frac{2a}{b} + \frac{b}{a} & 2 + \frac{b^2}{a^2} \\
 & 2 + \frac{b^2}{a^2} \\ \hline
\end{array}$$

5.
$$\begin{array}{l|l}
 & x^4 - 2x^3 + 2x^2 - x + \frac{1}{4}\ \left(x^2 - x + \frac{1}{2}\right. \\
 & x^4 \\ \hline
2x^2 - x & -2x^3 + 2x^2 \\
 & -2x^3 + x^2 \\ \hline
2x^2 - 2x + \frac{1}{2} & x^2 - x + \frac{1}{4} \\
 & x^2 - x + \frac{1}{4} \\ \hline
\end{array}$$

6.
$$\begin{array}{l|l}
 & x^4 + 2x^3 - x + \frac{1}{4}\ \left(x^2 + x - \frac{1}{2}\right. \\
 & x^4 \\ \hline
2x^2 + x & 2x^3 - x \\
 & 2x^3 + x^2 \\ \hline
2x^2 + 2x - \frac{1}{2} & -x^2 - x + \frac{1}{4} \\
 & -x^2 - x + \frac{1}{4} \\ \hline
\end{array}$$

7. $$4a^2 - 12ab + ab^2 + 9b^2 - \frac{3b^3}{2} + \frac{b^4}{16}\left(2a - 3b + \frac{b^2}{4}\right.$$

$$\begin{array}{l|l}
 & 4a^2 \\ \cline{2-2}
4a - 3b & -12ab + ab^2 + 9b^2 \\
 & -12ab + 9b^2 \\ \cline{2-2}
4a - 6b + \frac{b^2}{4} & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
 & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\ \cline{2-2}
\end{array}$$

8. $$x^4 + 8x^2 + 24 + \frac{16}{x^4} + \frac{32}{x^2}\left(x^2 + 4 + \frac{4}{x^2}\right.$$

$$\begin{array}{l|l}
 & x^4 \\ \cline{2-2}
2x^2 + 4 & 8x^2 + 24 \\
 & 8x^2 + 16 \\ \cline{2-2}
2x^2 + 8 + \frac{4}{x^2} & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
 & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\ \cline{2-2}
\end{array}$$

9. $$\frac{16}{9}a^6x^2 + \frac{16}{3}a^5x + 4a^4 - 2a^3x - 3a^2 + \frac{9}{16}\left(\frac{4a^3x}{3} + 2a^2 - \frac{3}{4}\right.$$

$$\begin{array}{l|l}
 & \frac{16}{9}a^6x^2 \\ \cline{2-2}
\frac{8a^3x}{3} + 2a^2 & \frac{16}{3}a^5x + 4a^4 \\
 & \frac{16}{3}a^5x + 4a^4 \\ \cline{2-2}
\frac{8a^3x}{3} + 4a^2 - \frac{3}{4} & -2a^3x - 3a^2 + \frac{9}{16} \\
 & -2a^3x - 3a^2 + \frac{9}{16} \\ \cline{2-2}
\end{array}$$

10.
$$\frac{1}{x^2}-\frac{4}{xy}+\frac{4}{y^2}+\frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}\left(\frac{1}{x}-\frac{2}{y}+\frac{3}{z}\right.$$

$$\begin{array}{l|l}
 & \frac{1}{x^2} \\ \hline
\frac{2}{x}-\frac{2}{y} & -\frac{4}{xy}+\frac{4}{y^2} \\
 & -\frac{4}{xy}+\frac{4}{y^2} \\ \hline
\frac{2}{x}-\frac{4}{y}+\frac{3}{z} & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\
 & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\ \hline
\end{array}$$

11.
$$36m^2-\frac{48m}{n}+\frac{16}{n^2}+\frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}\left(6m-\frac{4}{n}+\frac{p}{5}\right.$$

$$\begin{array}{l|l}
 & 36m^2 \\ \hline
12m-\frac{4}{n} & -\frac{48m}{n}+\frac{16}{n^2} \\
 & -\frac{48m}{n}+\frac{16}{n^2} \\ \hline
12m-\frac{8}{n}+\frac{p}{5} & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\
 & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\ \hline
\end{array}$$

12.
$$a^2b^2-6abcd+9c^2d^2+\frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}\left(ab-3cd+\frac{ef}{7}\right.$$

$$\begin{array}{l|l}
 & a^2b^2 \\ \hline
2ab-3cd & -6abcd+9c^2d^2 \\
 & -6abcd+9c^2d^2 \\ \hline
2ab-6cd+\frac{ef}{7} & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\
 & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\ \hline
\end{array}$$

13.

$$\frac{4x^2}{z^2}-\frac{12xy}{z^2}+\frac{9y^2}{z^2}+4-\frac{6y}{x}+\frac{z^2}{x^2}\left(\frac{2x}{z}-\frac{3y}{z}+\frac{z}{x}\right.$$

$$\begin{array}{l|l}
 & \frac{4x^2}{z^2} \\ \cline{2-2}
\frac{4x}{z}-\frac{3y}{z} & -\frac{12xy}{z^2}+\frac{9y^2}{z^2} \\
 & -\frac{12xy}{z^2}+\frac{9y^2}{z^2} \\ \cline{2-2}
\frac{4x}{z}-\frac{3y}{z}+\frac{z}{x} & 4-\frac{6y}{x}+\frac{z^2}{x^2} \\
 & 4-\frac{6y}{x}+\frac{z^2}{x^2} \\ \cline{2-2}
\end{array}$$

14.

$$\frac{4m^2}{n^2}-\frac{16m}{n}+4+\frac{24n}{m}+\frac{9n^2}{m^2}\left(\frac{2m}{n}-4-\frac{3n}{m}\right.$$

$$\begin{array}{l|l}
 & \frac{4m^2}{n^2} \\ \cline{2-2}
\frac{4m}{n}-4 & -\frac{16m}{n}+4 \\
 & -\frac{16m}{n}+16 \\ \cline{2-2}
\frac{4m}{n}-8-\frac{3n}{m} & -12+\frac{24n}{m}+\frac{9n^2}{m^2} \\
 & -12+\frac{24n}{m}+\frac{9n^2}{m^2} \\ \cline{2-2}
\end{array}$$

15.

$$\begin{array}{ll}
 & \frac{a^2}{9}-\frac{ab}{6}+\frac{b^2}{16}+\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}\Big(\frac{a}{3}-\frac{b}{4}+\frac{c}{5}-\frac{d}{2} \\
 & \frac{a^2}{9} \\ \hline
\frac{2a}{3}-\frac{b}{4} & -\frac{ab}{6}+\frac{b^2}{16} \\
 & -\frac{ab}{6}+\frac{b^2}{16} \\ \hline
\frac{2a}{3}-\frac{b}{2}+\frac{c}{5} & \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25} \\
 & \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25} \\ \hline
\frac{2a}{3}-\frac{b}{2}+\frac{2c}{5}-\frac{d}{2} & -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4} \\
 & -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4} \\ \hline
\end{array}$$

16.

$$\begin{array}{ll}
 & 49x^4-28x^3-17x^2+6x+\frac{9}{4}\Big(7x^2-2x-\frac{3}{2} \\
 & 49x^4 \\ \hline
14x^2-2x & -28x^3-17x^2 \\
 & -28x^3+4x^2 \\ \hline
14x^2-4x-\frac{3}{2} & -21x^2+6x+\frac{9}{4} \\
 & -21x^2+6x+\frac{9}{4} \\ \hline
\end{array}$$

17.

$$\begin{array}{ll}
 & 9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2\Big(3x^2-\frac{ax}{2}+bx \\
 & 9x^4 \\ \hline
6x^2-\frac{ax}{2} & -3ax^3+\frac{a^2x^2}{4} \\
 & -3ax^3+\frac{a^2x^2}{4} \\ \hline
6x^2-ax+bx & 6bx^3-abx^2+b^2x^2 \\
 & 6bx^3-abx^2+b^2x^2 \\ \hline
\end{array}$$

18.

$$\begin{array}{l|l}
 & 9x^4-2x^3-\frac{161}{9}x^2+2x+9\ \big(3x^2-\frac{x}{3}-3 \\
 & 9x^4 \\ \hline
6x^2-\frac{x}{3} & -2x^3-\frac{161}{9}x^2 \\
 & -2x^3+\frac{x^2}{9} \\ \hline
6x^2-\frac{2x}{3}-3 & -18x^2+2x+9 \\
 & -18x^2+2x+9 \\ \hline
\end{array}$$

LXXXIII.

1.

$$\begin{array}{l|l}
 & a^3-3a^2b+3ab^2-b^3\,(a-b \\
 & a^3 \\ \hline
3a^2-3ab+b^2 & -3a^2b+3ab^2-b^3 \\
 & -3a^2b+3ab^2-b^3 \\ \hline
\end{array}$$

2.

$$\begin{array}{l|l}
 & 8a^3+12a^2+6a+1\,(2a+1 \\
 & 8a^3 \\ \hline
12a^2+6a+1 & 12a^2+6a+1 \\
 & 12a^2+6a+1 \\ \hline
\end{array}$$

3.

$$\begin{array}{l|l}
 & a^3+24a^2b+192ab^2+512b^3\,(a+8b \\
 & a^3 \\ \hline
3a^2+24ab+64b^2 & 24a^2b+192ab^2+512b^3 \\
 & 24a^2b+192ab^2+512b^3 \\ \hline
\end{array}$$

4.

$$\begin{array}{l|l}
 & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3\,(a+b+c \\
 & a^3 \\ \hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\ \hline
3a^2+6ab+3b^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
\quad +3ac+3bc+c^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\ \hline
\end{array}$$

5.

$$\begin{array}{l|l}
 & x^3-3x^2y+3xy^2-y^3+3x^2z-6xyz+3y^2z+3xz^2-3yz^2+z^3\,(x-y+z \\
 & x^3 \\ \hline
3x^2-3xy+y^2 & -3x^2y+3xy^2-y^3 \\
 & -3x^2y+3xy^2-y^3 \\ \hline
3x^2-6xy+3y^2 & 3x^2z-6xyz+3y^2z+3xz^2-3yz^2+z^3 \\
\quad +3xz-3yz+z^2 & 3x^2z-6xyz+3y^2z+3xz^2-3yz^2+z^3 \\ \hline
\end{array}$$

6.

$$\begin{array}{l|l}
 & 27x^6-54x^5+63x^4-44x^3+21x^2-6x+1\,(3x^2-2x+1 \\
 & 27x^6 \\ \cline{2-2}
27x^4-18x^3+4x^2 & -54x^5+63x^4-44x^3 \\
 & -54x^5+36x^4-8x^3 \\ \cline{2-2}
27x^4-36x^3+21x^2-6x+1 & 27x^4-36x^3+21x^2-6x+1 \\
 & 27x^4-36x^3+21x^2-6x+1 \\ \cline{2-2}
\end{array}$$

7.

$$\begin{array}{l|l}
 & 1-3a+6a^2-7a^3+6a^4-3a^5+a^6\,(1-a+a^2 \\
 & 1 \\ \cline{2-2}
3-3a+a^2 & -3a+6a^2-7a^3 \\
 & -3a+3a^2-a^3 \\ \cline{2-2}
3-6a+6a^2-3a^3+a^4 & 3a^2-6a^3+6a^4-3a^5+a^6 \\
 & 3a^2-6a^3+6a^4-3a^5+a^6 \\ \cline{2-2}
\end{array}$$

8.

$$\begin{array}{l|l}
 & x^3-3x^2y+3xy^2-y^3+8z^3+6x^2z-12xyz+6y^2z+12xz^2-12yz^2 \\
 & x^3 \qquad\qquad (x-y+2z \\ \cline{2-2}
3x^2-3xy+y^2 & -3x^2y+3xy^2-y^3 \\
 & -3x^2y+3xy^2-y^3 \\ \cline{2-2}
3x^2-6xy+3y^2 & 6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3 \\
\quad +6xz-6yz+4z^2 & 6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3 \\ \cline{2-2}
\end{array}$$

9.

$$\begin{array}{l|l}
 & a^6-12a^5+54a^4-112a^3+108a^2-48a+8\,(a^2-4a+2 \\
 & a^6 \\ \cline{2-2}
3a^4-12a^3+16a^2 & -12a^5+54a^4-112a^3 \\
 & -12a^5+48a^4-64a^3 \\ \cline{2-2}
3a^4-24a^3+54a^2-24a+8 & 6a^4-48a^3+108a^2-48a+8 \\
 & 6a^4-48a^3+108a^2-48a+8 \\ \cline{2-2}
\end{array}$$

10. $8m^6 - 36m^5 + 66m^4 - 63m^3 + 33m^2 - 9m + 1\,(2m^2 - 3m + 1$

$8m^6$

$12m^4 - 18m^3 + 9m^2$ $\quad -36m^5 + 66m^4 - 63m^3$

$-36m^5 + 54m^4 - 27m^3$

$12m^4 - 36m^3 + 33m^2$ $\quad 12m^4 - 36m^3 + 33m^2 - 9m + 1$

$-9m + 1$ $\quad 12m^4 - 36m^3 + 33m^2 - 9m + 1$

11. $x^3 + 6x^2y + 12xy^2 + 8y^3 - 3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$

x^3 $\quad (x + 2y - z$

$3x^2 + 6xy + 4y^2$ $\quad 6x^2y + 12xy^2 + 8y^3$

$6x^2y + 12xy^2 + 8y^3$

$3x^2 + 12xy + 12y^2$ $\quad -3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$

$-3xz - 6yz + z^2$ $\quad -3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$

12. $8m^3 - 36m^2n + 54mn^2 - 27n^3 - 12m^2r + 36mnr - 27n^2r +$

$8m^3$ $\quad 6mr^2 - 9nr^2 - r^3\,(2m - 3n - r$

$12m^2 - 18mn + 9n^2$ $\quad -36m^2n + 54mn^2 - 27n^3$

$-36m^2n + 54mn^2 - 27n^3$

$12m^2 - 36mn + 27n^2 -$ $\quad -12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$

$6mr - 9nr + r^2$ $\quad -12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$

13. $m^3 + 3m^2 - 5 + \frac{3}{m^2} - \frac{1}{m^3}\,\left(m + 1 - \frac{1}{m}\right.$

m^3

$3m^2 + 3m + 1$ $\quad 3m^2 - 5 + \frac{3}{m^2}$

$3m^2 + 3m + 1$

$3m^2 + 6m - \frac{3}{m} + \frac{1}{m^2}$ $\quad -3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$

$-3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$

LXXXIV.

1

$$\begin{array}{ll}
 & 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4\,(4a^2 - 12ax + 9x^2 \\
 & 16a^4 \\
\cline{2-2}
8a^2 - 12ax & \quad -96a^3x + 216a^2x^2 \\
 & \quad -96a^3x + 144a^2x^2 \\
\cline{2-2}
8a^2 - 24ax + 9x^2 & \qquad\qquad 72a^2x^2 - 216ax^3 + 81x^4 \\
 & \qquad\qquad 72a^2x^2 - 216ax^3 + 81x^4 \\
\cline{2-2}
\end{array}$$

and the square root of $4a^2 - 12ax + 9x^2$ is $2a - 3x$.

2.

$$\begin{array}{ll}
 & 1 - 8a + 24a^2 - 32a^3 + 16a^4\,(1 - 4a + 4a^2 \\
 & 1 \\
\cline{2-2}
2 - 4a & \quad -8a + 24a^2 \\
 & \quad -8a + 16a^2 \\
\cline{2-2}
2 - 8a + 4a^2 & \qquad\qquad 8a^2 - 32a^3 + 16a^4 \\
 & \qquad\qquad 8a^2 - 32a^3 + 16a^4 \\
\cline{2-2}
\end{array}$$

and the square root of $1 - 4a + 4a^2$ is $1 - 2a$.

3.

$$\begin{array}{ll}
 & 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4\,(25 + 40x + 16x^2 \\
 & 625 \\
\cline{2-2}
50 + 40x & \quad 2000x + 2400x^2 \\
 & \quad 2000x + 1600x^2 \\
\cline{2-2}
50 + 80x + 16x^2 & \qquad\qquad 800x^2 + 1280x^3 + 256x^4 \\
 & \qquad\qquad 800x^2 + 1280x^3 + 256x^4 \\
\cline{2-2}
\end{array}$$

and the square root of $25 + 40x + 16x^2$ is $5 + 4x$.

4.

$$
\begin{array}{ll}
 & a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \,(a^3 - 3a^2b + 3ab^2 - b^3 \\
 & a^6 \\ \cline{2-2}
2a^3 - 3a^2b & -6a^5b + 15a^4b^2 \\
 & -6a^5b + 9a^4b^2 \\ \cline{2-2}
2a^3 - 6a^2b + 3ab^2 & 6a^4b^2 - 20a^3b^3 + 15a^2b^4 \\
 & 6a^4b^2 - 18a^3b^3 + 9a^2b^4 \\ \cline{2-2}
2a^3 - 6a^2b + 6ab^2 - b^3 & -2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6 \\
 & -2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6 \\ \cline{2-2}
\end{array}
$$

and the cube root of $a^3 - 3a^2b + 3ab^2 - b^3$ is $a - b$.

5.

$$
\begin{array}{ll}
 & x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \,(x^3 + 3x^2 + 3x + 1 \\
 & x^6 \\ \cline{2-2}
2x^3 + 3x^2 & 6x^5 + 15x^4 \\
 & 6x^5 + 9x^4 \\ \cline{2-2}
2x^3 + 6x^2 + 3x & 6x^4 + 20x^3 + 15x^2 \\
 & 6x^4 + 18x^3 + 9x^2 \\ \cline{2-2}
2x^3 + 6x^2 + 6x + 1 & 2x^3 + 6x^2 + 6x + 1 \\
 & 2x^3 + 6x^2 + 6x + 1 \\ \cline{2-2}
\end{array}
$$

and the cube root of $x^3 + 3x^2 + 3x + 1$ is $x + 1$.

6.

$$
\begin{array}{ll}
 & m^6 - 12m^5 + 60m^4 - 160m^3 + 240m^2 - 192m + 64 \,(m^3 - 6m^2 + 12m - 8 \\
 & m^6 \\ \cline{2-2}
2m^3 - 6m^2 & -12m^5 + 60m^4 \\
 & -12m^5 + 36m^4 \\ \cline{2-2}
2m^3 - 12m^2 + 12m & 24m^4 - 160m^3 + 240m^2 \\
 & 24m^4 - 144m^3 + 144m^2 \\ \cline{2-2}
2m^3 - 12m^2 + 24m - 8 & -16m^3 + 96m^2 - 192m + 64 \\
 & -16m^3 + 96m^2 - 192m + 64 \\ \cline{2-2}
\end{array}
$$

and the cube root of $m^3 - 6m^2 + 12m - 8$ is $m - 2$.

LXXXV.

1. $x=\pm 8$. 2. $x=\pm ab$. 3. $x^2=10000$; $x=\pm 100$.

4. $x^2=49$; $x=\pm 7$. 5. $3x^2=33$; $x^2=11$; $x=\pm\sqrt{11}$.

6. $x^2=64a^4c^6$, etc. 7. $4x^2-48=3x^2-12$; $x^2=36$, etc.

8. $250000-x^2=233359$; $x^2=16641$; $x=\pm 129$.

9. $8112=3x^2$; $x^2=2704$, etc.

10. $\frac{11x^2}{2}-18x+65=9x^2-18x+9$; $11x^2-18x^2=-112$; $7x^2=112$, etc.

11. $mx^2=q-n$; $x^2=\frac{q-n}{m}$, etc.

12. $x^2-ax+b=ax^2-ax$; $ax^2-x^2=b$; $x^2=\frac{b}{a-1}$, etc.

13. $180x^2-225=114x^2+171$; $66x^2=396$; $x^2=6$, etc.

14. $42x^2-126=35x^2-70$; $7x^2=56$; $x^2=8$; $x^2=4\times 2$; $x=\pm 2\sqrt{2}$.

LXXXVI.

1. $x^2+6x+9=81$; $x+3=\pm 9$, etc.

2. $x^2+12x+36=100$; $x+6=\pm 10$, etc.

3. $x^2+14x+49=64$; $x+7=\pm 8$, etc.

4. $x^2+46x+529=625$; $x+23=\pm 25$, etc.

5. $x^2+128x+4096=4489$; $x+64=\pm 67$, etc.

6. $x^2+8x+16=81$; $x+4=\pm 9$, etc.

7. $x^2+18x+81=324$; $x+9=\pm 18$, etc.

8. $x^2+16x+64=484$; $x+8=\pm 22$, etc.

LXXXVII.

1. $x^2-6x+9=16$; $x-3=\pm 4$, etc.

2. $x^2-4x+4=9$; $x-2=\pm 3$, etc.

3. $x^2-20x+100=121$; $x-10=\pm 11$, etc.

4. $x^2-2x+1=64$; $x-1=\pm 8$, etc.

5. $x^2-12x+36=4$; $x-6=\pm 2$, etc.

6. $x^2-14x+49=4$; $x-7=\pm 2$, etc.

7. $x^2-234x+13689=1$; $x-117=\pm 1$, etc.

8. $x^2-5x+6=15x+42$; $x^2-20x=36$; $x^2-20x+100=136$;
$x-10=\pm\sqrt{4\times 34}$; $x=10\pm 2\sqrt{34}$.

9. $3x^2-17x-2x^2-5x+120=0$; $x^2-22x+121=1$, etc.

10. $x^2-10x+25+x^2-14x+49=x^2-8x+46$; $x^2-16x=-28$, etc.

LXXXVIII.

1. $x^2+7x+\frac{49}{4}=\frac{169}{4}$; $x+\frac{7}{2}=\pm\frac{13}{2}$, etc.

2. $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

3. $x^2+9x+\frac{81}{4}=\frac{256}{4}$; $x+\frac{9}{2}=\pm\frac{16}{2}$, etc.

4. $x^2-13x+\frac{169}{4}=\frac{729}{4}$; $x-\frac{13}{2}=\pm\frac{27}{2}$, etc.

5. $x^2+x+\frac{1}{4}=\frac{9}{16}$; $x+\frac{1}{2}=\pm\frac{3}{4}$, etc.

6. $x^2-x+\frac{1}{4}=\frac{289}{4}$; $x-\frac{1}{2}=\pm\frac{17}{2}$, etc.

7. $x^2+37x+\frac{1369}{4}=\frac{16129}{4}$; $x+\frac{37}{2}=\pm\frac{127}{2}$, etc.

8. $x^2-x+\frac{1}{4}=\frac{225}{4}$; $x-\frac{1}{2}=\pm\frac{15}{2}$, etc.

9. $5x-x^2+2x^2-14x-10x+60=0$; $x^2-19x=-60$;

$x^2-19x+\frac{361}{4}=\frac{121}{4}$, etc.

10. $35x^2-312x+693-34x^2+21x+45=448$; $x^2-291x=-290$;

$x^2-291x+\frac{84681}{4}=\frac{83521}{4}$; $x-\frac{291}{2}=\pm\frac{289}{2}$, etc.

LXXXIX.

1. $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{36}{9}$; $x-\frac{1}{3}=\pm\frac{6}{3}$, etc.

2. $x^2+\frac{4}{5}x+\frac{4}{25}=\frac{1}{25}$; $x+\frac{2}{5}=\pm\frac{1}{5}$, etc.

3. $x^2-\frac{28x}{9}+\frac{196}{81}=\frac{169}{81}$; $x-\frac{14}{9}=\pm\frac{13}{9}$, etc.

4. $x^2-\frac{8}{11}x+\frac{16}{121}=\frac{49}{121}$; $x-\frac{4}{11}=\pm\frac{7}{11}$, etc.

5. $x^2+\frac{4}{35}x+\frac{4}{1225}=\frac{529}{1225}$; $x+\frac{2}{35}=\pm\frac{23}{35}$, etc.

6. $x^2-\frac{16}{5}x+\frac{64}{25}=\frac{144}{25}$; $x-\frac{8}{5}=\pm\frac{12}{5}$, etc.

7. $x^2-\frac{26}{3}x+\frac{169}{9}=\frac{121}{9}$; $x-\frac{13}{3}=\pm\frac{11}{3}$, etc.

8. $x^2 - \frac{4}{7}x + \frac{4}{49} = \frac{2209}{49}$; $x - \frac{2}{7} = \pm\frac{47}{7}$, etc.

XC.

1. $x^2 - \frac{1}{3}x + \frac{1}{36} = \frac{289}{36}$; $x - \frac{1}{6} = \pm\frac{17}{6}$, etc.

2. $x^2 - \frac{1}{5}x + \frac{1}{100} = \frac{9801}{100}$; $x - \frac{1}{10} = \pm\frac{99}{10}$, etc.

3. $x^2 + \frac{1}{2x} + \frac{1}{16} = \frac{625}{16}$; $x + \frac{1}{4} = \pm\frac{25}{4}$, etc.

4. $x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{1225}{16}$; $x + \frac{3}{4} = \pm\frac{35}{4}$, etc.

5. $x^2 - \frac{9}{5}x + \frac{81}{100} = \frac{1681}{100}$; $x - \frac{9}{10} = \pm\frac{41}{10}$, etc.

6. $x^2 - \frac{11}{2}x + \frac{121}{16} = \frac{25}{16}$; $x - \frac{11}{4} = \pm\frac{5}{4}$, etc.

7. $x^2 - \frac{15}{4}x + \frac{225}{64} = \frac{2401}{64}$; $x - \frac{15}{8} = \pm\frac{49}{8}$, etc.

8. $x^2 - \frac{23}{7}x + \frac{529}{196} = \frac{676}{196}$; $x - \frac{23}{14} = \pm\frac{26}{14}$, etc.

XCI.

1. $x^2 + 2ax + a^2 = 2a^2$; $x + a = \pm\sqrt{2}.a$, etc.

2. $x^2 - 4ax + 4a^2 = 11a^2$; $x - 2a = \pm\sqrt{11}.a$, etc.

3. $x^2 + 3mx + \frac{9m^2}{4} = 4m^2$; $x + \frac{3m}{2} = \pm 2m$, etc.

4. $x^2 - \frac{5nx}{2} + \frac{25n^2}{16} = \frac{49n^2}{16}$; $x - \frac{5n}{4} = \pm\frac{7n}{4}$, etc.

5. $x^2+(a-1)x+\frac{(a-1)^2}{4}=\frac{a^2+2a+1}{4}$; $x+\frac{a-1}{2}=\pm\frac{a+1}{2}$, etc.

6. $x^2+(a-b)x+\frac{(a-b)^2}{4}=\frac{(a+b)^2}{4}$; $x+\frac{a-b}{2}=\pm\frac{a+b}{2}$, etc.

7. $\frac{a^2}{(x+a)^2}=\frac{b^2}{(x-a)^2}$; and, taking the square root of each side,

$\frac{a}{x+a}=\pm\frac{b}{x-a}$, etc.

8. $acx^2-adx+bcx=bd$; $x^2+\frac{bc-ad}{ac}x=\frac{bd}{ac}$;

$x^2+\frac{bc-ad}{ac}x+\frac{(bc-ad)^2}{4a^2c^2}=\frac{(bc+ad)^2}{4a^2c^2}$;

$x+\frac{bc-ad}{2ac}=\pm\frac{bc+ad}{2ac}$; $x=\frac{2ad}{ac}$ or $-\frac{2bc}{2ac}$, etc.

9. $(a+b)x^2-cx=\frac{ac}{a+b}$; $x^2-\frac{cx}{a+b}=\frac{ac}{(a+b)^2}$;

$x^2-\frac{cx}{a+b}+\frac{c^2}{4(a+b)^2}=\frac{c^2+4ac}{4(a+b)^2}$; $x-\frac{c}{2(a+b)}=\pm\frac{\sqrt{c^2+4ac}}{2(a+b)}$, etc.

10. $x^2-\frac{2b^2x}{ac}=-\frac{b^4}{a^2c^2}$; $x^2-\frac{2b^2x}{ac}+\frac{b^4}{a^2c^2}=0$; $x-\frac{b^2}{ac}=0$; $x=\frac{b^2}{ac}$.

11. $x^2+\frac{3a^2+b^2}{abc}x=\frac{6a^2+ab-2b^2}{abc^2}$;

$x^2+\frac{3a^2+b^2}{abc}x+\frac{(3a^2+b^2)^2}{4a^2b^2c^2}=\frac{24a^3b+4a^2b^2-8ab^3+9a^4+6a^2b^2+b^4}{4a^2b^2c^2}$

$x+\frac{3a^2+b^2}{2abc}=\pm\frac{3a^2+4ab-b^2}{2abc}$, etc.

12. $x^2+\frac{4a^2c^2+4abd^2}{4a^2-9cd^2}x=-\frac{a^2c^4+2abc^2d^2+b^2d^4}{4a^2-9cd^2}$;

$$x^2+\frac{4a^2c^2+4abd^2}{4a^2-9cd^2}x+\frac{(2a^2c^2+2abd^2)^2}{(4a^2-9cd^2)^2}=\frac{9cd^2(a^2c^4+2abc^2d^2+b^2d^4)}{(4a^2-9cd^2)^2};$$

$$x+\frac{2a^2c^2+2abd^2}{4a^2-9cd^2}=\pm\frac{3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{-2a(ac^2+bd^2)\pm3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{(-2a\pm3d\sqrt{c})(ac^2+bd^2)}{(2a+3d\sqrt{c})(2a-3d\sqrt{c})}, \text{ etc.}$$

XCII.

1. $x^2-7x=8$; $x^2-7x+\frac{49}{4}=\frac{81}{4}$; $x-\frac{7}{2}=\pm\frac{9}{2}$, etc.

2. $x^2-5x=6$; $x^2-5x+\frac{25}{4}=\frac{49}{4}$; $x-\frac{5}{2}=\pm\frac{7}{2}$, etc.

3. $x^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

4. $x^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

5. $x^2+7x=18$; $x^2+7x+\frac{49}{4}=\frac{121}{4}$; $x+\frac{7}{2}=\pm\frac{11}{2}$, etc.

6. $4x^2-12x-12+x=22x-66$; $4x^2-33x=-54$; $x^2-\frac{33x}{4}=-\frac{54}{4}$;

$$x^2-\frac{33x}{4}+\frac{1089}{64}=\frac{225}{64};\ x-\frac{33}{8}=\pm\frac{15}{8}, \text{ etc.}$$

7. $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

8. $10x^2-30x-6x+6=7x^2-27x+18$; $3x^2-9x=12$; $x^2-3x=4$, etc.

9. $x^2-6x=16$; $x^2-6x+9=25$; $x-3=\pm 5$, etc.

10. $8x^2+20x-x^2+9=4x^2+22x+30$; $3x^2-2x=21$;

$$x^2-\frac{2x}{3}=7\,;\ x^2-\frac{2x}{3}+\frac{1}{9}=\frac{64}{9}\,;\ x-\frac{1}{3}=\pm\frac{8}{3}\text{, etc.}$$

11. $8x^2+12x-x^2+49=4x^2+34x+42$; $3x^2-22x=-7$;

$$x^2-\frac{22x}{3}=-\frac{7}{3}\,;\ x^2-\frac{22x}{3}+\frac{121}{9}=\frac{100}{9}\,;\ x-\frac{11}{3}=\pm\frac{10}{3}\text{, etc.}$$

12. $x^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

13. $x^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

14. $12x^2-8x+177=192$; $x^2-\frac{2x}{3}=\frac{5}{4}$; $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{49}{36}$, etc.

15. $3x^2-169=26x$; $x^2-\frac{26x}{3}=\frac{169}{3}$; $x^2-\frac{26x}{3}+\frac{169}{9}=\frac{676}{9}$, etc.

16. $x^2-9x=-20$; $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

17. $6x^2-28x-48-300+20x=7x^2-56x+84$; $x^2-48x=-432$, etc.

18. $x^2-8x=-12$; $x^2-8x+16=4$; $x-4=\pm 2$, etc.

19. $9x^2-90x+125-54x^2=9x^2-75x$; $54x^2+15x=125$;

$$x^2+\frac{15x}{54}=\frac{125}{54}\,;\ x^2+\frac{15x}{54}+\frac{225}{11664}=\frac{27225}{11664}\,;\ x+\frac{15}{108}=\pm\frac{165}{108}\text{, etc.}$$

20. $7x^2+35x-8x^2+20x=6x^2+16x-70$; $7x^2-39x=70$;

$$x^2-\frac{39x}{7}=10\,;\ x^2-\frac{39x}{7}+\frac{1521}{196}=\frac{3481}{196}\,;\ x-\frac{39}{14}=\pm\frac{59}{14}\text{, etc.}$$

21. $8x^2-20x+6x^2+16x-70=7x^2+35x$; $7x^2-39x=70$, etc.

22. $x^2-6x+9+4x=44$; $x^2-2x=35$, etc.

23. $x^2+11x=7x^2-9-4x$; $6x^2-15x=9$; $2x^2-5x=3$; $x^2-\frac{5x}{2}=\frac{3}{2}$;
$x^2-\frac{5x}{2}+\frac{25}{16}=\frac{49}{16}$; $x-\frac{5}{4}=\pm\frac{7}{4}$, etc.

24. $x^2+\frac{x}{6}=\frac{2}{6}$; $x^2+\frac{x}{6}+\frac{1}{144}=\frac{49}{144}$; $x+\frac{1}{12}=\pm\frac{7}{12}$, etc.

25. $x^2-\frac{x}{2}+\frac{1}{16}=\frac{25}{144}$; $x-\frac{1}{4}=\pm\frac{5}{12}$, etc.

26. $x^2-x+\frac{1}{4}=\frac{841}{4}$; $x-\frac{1}{2}=\pm\frac{29}{2}$, etc.

27. $6x+2x+2=3x^2+3x$; $3x^2-5x=2$; $x^2-\frac{5x}{3}=\frac{2}{3}$, etc.

28. $4x^2-33=x$; $x^2-\frac{x}{4}=\frac{33}{4}$; $x^2-\frac{x}{4}+\frac{1}{64}=\frac{529}{64}$, etc.

29. $2x^2=3x^2-3x+2x^2-4x+2$; $x^2-\frac{7x}{3}=-\frac{2}{3}$, etc.

30. $x^2-\frac{7x}{15}=\frac{46}{15}$; $x^2-\frac{7x}{15}+\frac{49}{900}=\frac{2809}{900}$; $x-\frac{7}{30}=\pm\frac{53}{30}$, etc.

31. $5x+10-10x+20=3x^2-12$; $x^2+\frac{5x}{3}=14$, etc.

32. $4x^2-100+40x-4x^2=75x-15x^2$; $15x^2-35x=100$;
$x^2-\frac{7x}{3}=\frac{20}{3}$; $x^2-\frac{7x}{3}+\frac{49}{36}=\frac{289}{36}$, etc.

33. $90x-126+18x=22x^2$; $x^2-\frac{54x}{11}=-\frac{63}{11}$; $x^2-\frac{54x}{11}+\frac{729}{121}=\frac{36}{121}$, etc.

34. $3x^2-5x=7x+420$; $x^2-4x=140$, etc.

35. $\frac{48-12x+40-8x}{20-9x+x^2}=\frac{32}{x+2}$; $\frac{22-5x}{20-9x+x^2}=\frac{8}{x+2}$;
$44+12x-5x^2=160-72x+8x^2$; $13x^2-84x=-116$;
$x^2-\frac{84x}{13}=-\frac{116}{13}$; $x^2-\frac{84x}{13}+\frac{1764}{169}=\frac{256}{169}$, etc.

36. $10x^2+490-140x+10x^2=203x-29x^2$; $49x^2-343x=-490$; $x^2-7x=-10$, etc.

37. $x^2+(a+b)x+\frac{(a+b)^2}{4}=\frac{(a-b)^2}{4}$; $x+\frac{a+b}{2}=\pm\frac{a-b}{2}$, etc.

38. $x^2-(b-a)x+\frac{(b-a)^2}{4}=\frac{(b+a)^2}{4}$; $x-\frac{b-a}{2}=\pm\frac{b+a}{2}$, etc.

39. $x^2-2ax+a^2=b^2$; $x-a=\pm b$, etc.

40. $x^2-(a^2-a^3)x+\frac{(a^2-a^3)^2}{4}=\frac{(a^2+a^3)^2}{4}$; $x-\frac{a^2-a^3}{2}=\pm\frac{a^2+a^3}{2}$, etc.

41. $x^2+\frac{ax}{b}+\frac{a^2}{4b^2}=\frac{9a^2}{4b^2}$; $x+\frac{a}{2b}=\pm\frac{3a}{2b}$, etc.

42. $x^2-\frac{a^2+b^2}{ab}x+\frac{(a^2+b^2)^2}{4a^2b^2}=\frac{(a^2-b^2)^2}{4a^2b^2}$; $x-\frac{a^2+b^2}{2ab}=\pm\frac{a^2-b^2}{2ab}$.

XCIII.

1. $\left.\begin{array}{l} x^2+2xy+y^2=1600 \\ 4xy \quad\quad\; =1200 \end{array}\right\}$, $x^2-2xy+y^2=400$, $\left\{\begin{array}{l} x+y=\quad 40 \\ x-y=\pm 20 \end{array}\right.$, etc.

2. $\left.\begin{array}{l} x^2+2xy+y^2=169 \\ 4xy \quad\quad\; =144 \end{array}\right\}$, $x^2-2xy+y^2=25$, $\left\{\begin{array}{l} x+y=\quad 13 \\ x-y=\pm\ 5 \end{array}\right.$, etc.

3. $\left.\begin{array}{l} x^2+2xy+y^2=841 \\ 4xy \quad\quad\; =400 \end{array}\right\}$, $x^2-2xy+y^2=441$, $\left\{\begin{array}{l} x+y=\quad 29 \\ x-y=\pm 21 \end{array}\right.$, etc.

4. $\left.\begin{array}{l} x^2-2xy+y^2=361 \\ 4xy \quad\quad\; =264 \end{array}\right\}$, $x^2+2xy+y^2=625$, $\left\{\begin{array}{l} x-y=\quad 19 \\ x+y=\pm 25 \end{array}\right.$, etc.

5. $\left.\begin{array}{l} x^2-2xy+y^2=2025 \\ 4xy \quad\quad\; =1000 \end{array}\right\}$, $x^2+2xy+y^2=3025$, $\left\{\begin{array}{l} x-y=\quad 45 \\ x+y=\pm 55 \end{array}\right.$, etc.

6. $\left.\begin{array}{l} x^2-2xy+y^2=9801 \\ 4xy \quad\quad\; =\ 400 \end{array}\right\}$, $x^2+2xy+y^2=10201$, $\left\{\begin{array}{l} x-y=\quad 99 \\ x+y=\pm 101 \end{array}\right.$, etc.

XCIV.

1. $\left.\begin{array}{l} x^2-2xy+y^2=16 \\ x^2 \qquad\quad +y^2=40 \end{array}\right\}$; $2xy=24$; $x^2+2xy+y^2=64$, etc.

2. $\left.\begin{array}{l} x^2-2xy+y^2=100 \\ x^2 \qquad\quad +y^2=178 \end{array}\right\}$; $2xy=78$; $x^2+2xy+y^2=256$, etc.

3. $\left.\begin{array}{l} x^2-2xy+y^2=196 \\ x^2 \qquad\quad +y^2=436 \end{array}\right\}$; $2xy=240$; $x^2+2xy+y^2=676$, etc.

4. $\left.\begin{array}{l} x^2+2xy+y^2=64 \\ x^2 \qquad\quad +y^2=32 \end{array}\right\}$; $2xy=32$; $x^2-2xy+y^2=0$, etc.

5. $\left.\begin{array}{l} x^2+2xy+y^2=144 \\ x^2 \qquad\quad +y^2=104 \end{array}\right\}$; $2xy=40$; $x^2-2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+2xy+y^2=2401 \\ x^2 \qquad\quad +y^2=1681 \end{array}\right\}$; $2xy=720$; $x^2-2xy+y^2=961$, etc.

XCV.

1. $\left.\begin{array}{l} x^2-\ \ xy+y^2=13 \\ x^2+2xy+y^2=49 \end{array}\right\}$; $3xy=36$; $x^2-2xy+y^2=1$, etc.

2. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 31 \\ x^2+2xy+y^2=121 \end{array}\right\}$; $3xy=90$; $x^2-2xy+y^2=1$, etc.

3. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 84 \\ x^2+2xy+y^2=144 \end{array}\right\}$; $3xy=60$; $x^2-2xy+y^2=64$, etc.

4. $\left.\begin{array}{l} x^2+\ \ xy+y^2=28 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=24$; $x^2+2xy+y^2=36$, etc.

5. $\left.\begin{array}{l} x^2+\ \ xy+y^2=49 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=45$; $x^2+2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+\ \ xy+y^2=93 \\ x^2-2xy+y^2=\ \ 9 \end{array}\right\}$; $3xy=84$; $x^2+2xy+y^2=121$, etc.

XCVI.

1. $$\left.\begin{array}{l}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{81}{400}\\ \frac{1}{x^2}\qquad\ +\frac{1}{y^2}=\frac{41}{400}\end{array}\right\};\ \frac{2}{xy}=\frac{40}{400};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400},\text{ etc.}$$

2. $$\left.\begin{array}{l}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{9}{16}\\ \frac{1}{x^2}\qquad\ +\frac{1}{y^2}=\frac{5}{16}\end{array}\right\};\ \frac{2}{xy}=\frac{4}{16};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{16},\text{ etc.}$$

3. $$\left.\begin{array}{l}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=25\\ \frac{1}{x^2}\qquad\ +\frac{1}{y^2}=13\end{array}\right\};\ \frac{2}{xy}=12;\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1,\text{ etc.}$$

4. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{12}$, etc.

5. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{2}$, etc.

6. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=7$, etc.

XCVII.

1. $x-y=1$; $x^2-2xy+y^2=1$; $3xy=36$, etc.

2. $x^2+12xy+36y^2=576$; $x+6y=\pm 24$.

Now, from the first equation, $x(x+6y)=144$, $\therefore\ \pm 24x=144$, etc.

3. $x^2+2xy+y^2=441$; $x+y=\pm 21$.

Now, from the first equation, $x(x+y)=210$, $\therefore\ \pm 21x=210$, etc.

4. $x^2+2xy+y^2=100$; and $x^2-2xy+y^2=36$;

hence $x+y=\pm 10$, and $x-y=\pm 6$, etc.

5. $x+y=8$; $x^2+2xy+y^2=64$; $3xy=45$, etc.

6. $4x=10+5y$; put $\dfrac{10+5y}{4}$ for x in the first equation;

then $\dfrac{(10+5y)^2}{4}+\dfrac{90y+45y^2}{4}=100$; $70y^2+190y=660$;

$7y^2+19y=66$; whence $y=2$ or $-\dfrac{33}{7}$, etc.

7. Put $y=mx$. Then $\dfrac{x^2+mx^2+m^2x^2}{3m^2x^2-5mx^2}=\dfrac{39}{25}$; $\dfrac{1+m+m^2}{3m^2-5m}=\dfrac{39}{25}$;

$25+25m+25m^2=117m^2-195m$; $92m^2-220m=25$;

hence we get $m=\dfrac{5}{2}$ or $-\dfrac{5}{46}$. Taking the value $\dfrac{5}{2}$, we get

$x^2+\dfrac{5x^2}{2}+\dfrac{25x^2}{4}=39$; $39x^2=39\times 4$; $x=\pm 2$, etc.

8. Put $y=mx$. Then $\dfrac{x^2+mx^2}{mx^2-m^2x^2}=\dfrac{66}{5}$; $\dfrac{1+m}{m-m^2}=\dfrac{66}{5}$;

$5+5m=66m-66m^2$; $m^2-\dfrac{61m}{66}=-\dfrac{5}{66}$; hence we get

$m=\dfrac{1}{11}$, or $\dfrac{5}{6}$. Taking the value $\dfrac{5}{6}$, $x^2+\dfrac{5x^2}{6}=66$,

whence $x^2=36$, $x=\pm 6$, etc.

9. Put $y=mx$. Then $\dfrac{3+4m}{5m+2m^2}=\dfrac{20}{12}$; whence $m=\dfrac{1}{2}$ or $-\dfrac{3}{10}$.

Taking the value $\frac{1}{2}$, we get $3x^2+2x^2=20$; $x=\pm 2$, etc.

10. Put $y=mx$. Then $\dfrac{x^2-mx^2+m^2x^2}{3x^2+13mx^2+8m^2x^2}=\dfrac{7}{162}$; $\dfrac{1-m+m^2}{3+13m+8m^2}=\dfrac{7}{162}$

$162-162m+162m^2=21+91m+56m^2$; $106m^2-253m=-141$;

$m^2-\dfrac{253}{136}m=-\dfrac{141}{106}$; $m-\dfrac{253}{212}=\pm\dfrac{65}{212}$; $m=\dfrac{3}{2}$ or $-\dfrac{47}{53}$.

Taking the value $\dfrac{3}{2}$, we get $x^2-\dfrac{3x^2}{2}+\dfrac{9x^2}{4}=7$; $x=\pm 2$, etc.

11. Put $y = mx$. Then $\frac{1-m}{m+m^2} = \frac{35}{18}$; $35m^2 + 53m = 18$;

$m + \frac{53}{70} = \pm\frac{73}{70}$; $m = \frac{2}{7}$ or $-\frac{9}{10}$.

Taking the value $\frac{2}{7}$ we get $x = \pm 7$, etc.

12. $x = \frac{29-7y}{5}$; putting this for x in the first equation, we get

$\frac{2523 - 1218y + 147y^2}{25} + \frac{116y - 28y^2}{5} + 5y^2 = 71$;

$132y^2 - 638y = -748$; $y = 2$ or $\frac{187}{66}$, etc.

13. $x + y = 22$; $x^2 + 2xy + y^2 = 484$; $3xy = 360$; $xy = 120$;

$x^2 - 2xy + y^2 = 4$; $x - y = \pm 2$, etc.

14. Subtract the second equation from the first; $x^2 + 2xy + y^2 = 169$;

$x + y = \pm 13$. Take the positive value, and for y put $13 - x$ in the first equation; $x^2 + 117x - 9x^2 = 340$; $x^2 - \frac{117}{8}x = -\frac{340}{8}$;

$x^2 - \frac{117}{8}x + \frac{13689}{256} = \frac{2809}{256}$; $x - \frac{117}{16} = \pm\frac{53}{16}$, etc.

15. $\left.\begin{array}{r} x^2 + y^2 = 225 \\ 2xy = 216 \end{array}\right\}$, $\left.\begin{array}{r} x^2 + y^2 + 2xy = 441 \\ x^2 + y^2 - 2xy = 9 \end{array}\right\}$, $\left.\begin{array}{r} x + y = \pm 21 \\ x - y = \pm 3 \end{array}\right\}$, etc.

XCVIII.

1. Let x be the number; then $\frac{x}{2} \times \frac{x}{3} = 864$; $x^2 = 5184$, etc.

2. Let x be the number; then $\left(\frac{x}{7} \times \frac{x}{8}\right) \div 3 = 298\frac{2}{3}$; $\frac{x^2}{56} = 896$, etc.

3. Let x be the number; then $(94 - x)(94 + x) = 8512$, etc.

4. Let x be the greater; then $\frac{750}{x}$ is the less; and $x \times \frac{x}{750} = 3\frac{1}{3}$, etc.

5. Let x and y be the numbers; then $x^2 + y^2 = 13001$ and $x^2 - y^2 = 1449$; adding, $2x^2 = 14450$, etc.

6. Let x be the greater; then $\frac{377}{x}$ is the less; and $x - 21 = 21 - \frac{377}{x}$;

$x^2 - 42x = -377$; $x^2 - 42x + 441 = 64$, etc.

7. Let x be the number; then $\frac{x}{2} \times \frac{x}{3} \times \frac{x}{4} \times \frac{x}{5} = 6750$; $x^4 = 810000$;

$x^2 = 900$; $x = 30$.

8. Let x be the number; then $\frac{11500}{x} = x + \frac{51}{x}$; $11500 = x^2 + 51$, etc.

9. Let x be the number; then $(x+20)^2 + 2(x-10)^2 = 17475$;

$x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; $3x^2 = 16875$, etc.

10. Let x and $26 - x$ be the numbers; then $x^2 + (26 - x)^2 = 436$;

$2x^2 - 52x + 676 = 436$; $x^2 - 26x = -120$, etc.

11. Let $x + 17$ and x be the numbers; then $(x+17)^2 + x^2 = 325$, etc.

12. Let x and $\frac{255}{x}$ be the numbers; then $x^2 + \frac{65025}{x^2} = 514$;

$x^4 - 514x^2 = -65025$; $x^4 - 514x^2 + 66049 = 1024$;

$x^2 - 257 = \pm 32$; $x^2 = 289$ or 225; $x = 17$ or 15.

13. Let x and $16 - x$ be the parts; then $x^2 + (16 - x)^2 + (16 - x)x = 208$

$x^2 - 16x + 256 = 208$; $x^2 - 16x = -48$, etc.

14. Let x^2 be the number; then $x^2 + x = 1332$; $x^2 + x + \frac{1}{4} = \frac{532}{4}$

$x + \frac{1}{2} = \pm\frac{73}{2}$; $x = 36$; $x^2 = 1296$.

15. Let x^2 be the number; then $x^2-x=\frac{195}{4}$, etc.

16. Let x^2 be the number; then $x^2-x=2550$, etc.

17. Let x and $\frac{24}{x}$ be the numbers; then $\left(x+\frac{24}{x}\right)\left(x-\frac{24}{x}\right)=20$;

$x^2-\frac{576}{x^2}=20$; $x^4-20x^2=576$; $x^4-20x^2+100=676$;

$x^2-10=\pm 26$; taking the positive value $x=6$, etc.

18. Let x be the greater, y the less; then $x\,(x+y)=204$ and

$y\,(x-y)=35$, or, $x^2+xy=204$ and $xy-y^2=35$.

Put $y=mx$. Then $\frac{x^2+mx^2}{mx^2-m^2x^2}=\frac{204}{35}$; hence one value of m is $\frac{5}{12}$

and $x^2+\frac{5x^2}{12}=204$; $17x^2=2448$; $x^2=144$, etc.

19. Let x and $x-5$ be the numbers; then $x\,(2x-5)=228$, etc.

20. Let $x-1$, x, $x+1$ be the numbers; then $(x-1)\,x\,(x+1)=3x$;

$x^2-1=3$; $x^2=4$; $x=2$, etc.

21. Let x and $x+1$ be the numbers; then $(x+1)^2-x^2=15$, etc.

22. Let x and $x+1$ be the numbers; then $x^2+(x+1)^2=481$, etc.

23. Let $x-1$, x and $x+1$ be the numbers; then $(x-1)^2+x^2+(x+1)^2$

$=365$, etc.

24. Let x be the number; then $\frac{60}{x}=\frac{60}{x+3}+1$;

$60x+180=60x+x^2+3x$; $x^2+3x=180$, etc.

25. Let x be the number; then $\frac{80}{x}=\frac{80}{x+4}+1$, etc.

26. Let x be the price of each piece in shillings; then $675=48\times\frac{675}{x}-x$; $675x=32400-x^2$, etc.

27. Let x be the number; then $\frac{180}{x}=\frac{180}{x+3}+3$, etc.

28. Let x be the number of miles; then $\frac{108}{x}=\frac{108}{x+3}+6$, etc.

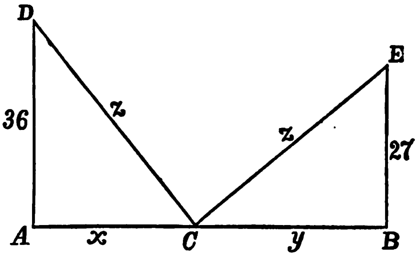
29. Let x be the number of sheep; then $\frac{60}{x}=\frac{54}{x-15}-\frac{1}{10}$, etc.

30. Let x and $x+2$ be the number of hours; then $\frac{1}{x}+\frac{1}{x+2}=\frac{1}{2\frac{11}{12}}$, etc.

31. Let x be the number of yards in the breadth; then $x(x+1)=10100$, etc.

32. Let $10x+y$ be the number; then $x=2y$ and $(10x+y)(10y+x)=2268$; hence $21y\times 12y=2268$; $y^2=9$, etc.

33. Let x and y be the distances in feet of the ladder from the bottoms of the houses; z the length of the ladder in feet.



Then since DCE is a right angle,

DCA and ECB together make a right angle. Eucl. i. 13.

But DCA and ADC together make a right angle. Eucl. i. 32.

$\therefore$ angle $ECB=$ angle ADC.

Hence triangles ACD and BEC are equal in all respects. Eucl. i. 26.

$\therefore$ $x=27$, and $y=36$; and $x+y=63$.

Also $z^2=(27)^2+(36)^2$. Eucl. i. 47; and $\therefore$ $z=45$.

34. Let x and y be the length and width in feet.

$$\left.\begin{aligned}&\text{Then } \frac{7x}{8}\times\frac{15y}{16}=xy-\frac{23}{4}\\ &\text{and } 2\left(\frac{7x}{8}+\frac{15y}{16}\right)=2(x+y)-\frac{17}{4}\end{aligned}\right\};$$

$$\left.\begin{aligned}105xy&=128xy-736\\ 14x+15y&=16x+16y-34\end{aligned}\right\};\quad \left.\begin{aligned}xy&=32\\ 2x+y&=34\end{aligned}\right\};\quad 34x-2x^2=32, \text{ etc.}$$

35. Let $10x+y$ be the number. Since the number is less than 50, and the difference of the digits 4, y is the greater of the digits.

$$\left.\begin{aligned}&\text{Then } y-x=4\\ &(10y+x)^2-(10x+y)^2=3960\end{aligned}\right\}, \text{ etc.}$$

36. Let x be the number of rows, $\frac{10000}{x}$ is then the number of trees in each row.

$$\text{Then } (x-20)\left(\frac{10000}{x}+25\right)=10000$$

$$\text{or, } 10000x-200000+25x^2-500x=10000x$$

$$\text{or, } x^2-20x=8000, \text{ etc.}$$

37. Let x be the number of men in the regiment; y the number in the side of the square at the first attempt.

Then $y^2=x-39$ and $(y+1)^2=x+50$.

Hence $(y+1)^2-y^2=89$; whence $2y=88$, $y=44$, etc.

XCIX.

1. $x=\frac{29-7y}{5}=5-y+\frac{4-2y}{5}$; let $\frac{4-2y}{5}=m$

$y=\frac{4-5m}{2}=2-2m-\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=2-4n-n=2-5n$

and $x=5-y+m=5-2+5n+2n=3+7n$

when $n=0$; $y=2$ and $x=3$.

2. $x=\frac{92-19y}{7}=13-2y-\frac{5y-1}{7}$; let $\frac{5y-1}{7}=m$

$y=\frac{7m+1}{5}=m+\frac{2m+1}{5}$; let $\frac{2m+1}{5}=n$

$m=\frac{5n-1}{2}=2n+\frac{n-1}{2}$; let $\frac{n-1}{2}=p$

$n=2p+1$

Then $y=m+n=2n+p+n=4p+2+p+2p+1=7p+3$

$x=13-14p-6-4p-2-p=5-19p$

when $p=0$, $x=5$ and $y=3$.

3. $x=\frac{1170-19y}{13}=90-y-\frac{6y}{13}$; let $\frac{6y}{13}=m$

$y=\frac{13m}{6}=2m+\frac{m}{6}$; let $\frac{m}{6}=n$; $m=6n$

Then $y=13n$ and $x=90-19n$

when $n=0$, $y=0$ and $x=90$

$n=1$, $y=13$ and $x=71$, etc.

4. $x=\frac{26-5y}{3}=8-y-\frac{2y-2}{3}$; let $\frac{2y-2}{3}=m$

$y=\frac{3m+2}{2}=m+1+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=2n+1+n=3n+1$

$x=8-3n-1-2n=7-5n$

when $n=0$, $y=1$, $x=7$

$n=1$, $y=4$, $x=2$.

5. $5y=14x-7$; $y=2x-1+\frac{4x-2}{5}$; let $\frac{4x-2}{5}=m$

$x=\frac{5m+2}{4}=m+\frac{m+2}{4}$; let $\frac{m+2}{4}=n$; $m=4n-2$

Then $x=4n-2+n=5n-2$

$y=10n-4-1+4n-2=14n-7$, etc.

6. $x=\dfrac{1031-15y}{11}=93-y-\dfrac{4y-8}{11}$; let $\dfrac{4y-8}{11}=m$

$y=\dfrac{11m+8}{4}=2m+2+\dfrac{3m}{4}$; let $\dfrac{3m}{4}=n$

$m=\dfrac{4n}{3}=n+\dfrac{n}{3}$; let $\dfrac{n}{3}=p$; $n=3p$

Then $y=8p+2+3p=11p+2$

$x=93-11p-2-4p=91-15p$, etc.

7. $y=\dfrac{308-11x}{7}=44-x-\dfrac{4x}{7}$; let $\dfrac{4x}{7}=m$

$x=\dfrac{7m}{4}=m+\dfrac{3m}{4}$; let $\dfrac{3m}{4}=n$

$m=\dfrac{4n}{3}=n+\dfrac{n}{3}$; let $\dfrac{n}{3}=p$; $n=3p$

Then $x=m+n=3p+p+3p=7p$

$y=44-x-m=44-7p-4p=44-11p$, etc.

8. $x=\dfrac{23+19y}{4}=5+4y+\dfrac{3y+3}{4}$; let $\dfrac{3y+3}{4}=m$

$y=\dfrac{4m-3}{3}=m-1+\dfrac{m}{3}$; let $\dfrac{m}{3}=n$; $m=3n$

Then $y=3n-1+n=4n-1$

$x=5+16n-4+3n=19n+1$, etc.

9. $y=\dfrac{20x-683}{9}=2x-75+\dfrac{2x-8}{9}$; let $\dfrac{2x-8}{9}=m$

$x=\dfrac{9m+8}{2}=4m+4+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $x=8n+4+n=9n+4$

$y=18n+8-75+2n=20n-67$, etc.

10. $x=\dfrac{383-7y}{3}=127-2y-\dfrac{y-2}{3}$; let $\dfrac{y-2}{3}=m$

Then $y=3m+2$

$x=127-6m-4-m=123-7m$. etc.

11. $y=\frac{54-27x}{4}=13-6x-\frac{3x-2}{4}$; let $\frac{3x-2}{4}=m$

$x=\frac{4m+2}{3}=m+\frac{m+2}{3}$; let $\frac{m+2}{3}=n$; $m=3n-2$

Then $x=3n-2+n=4n-2$

$y=13-24n+12-3n+2=27-27n$, etc.

12. $x=\frac{653-9y}{7}=93-y-\frac{2y-2}{7}$; let $\frac{2y-2}{7}=m$

$y=\frac{7m+2}{2}=3m+1+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=6n+1+n=7n+1$

$x=93-7n-1-2n=92-9n$, etc.

13. $\frac{x}{7}+\frac{y}{9}=\frac{57}{63}$; $9x+7y=57$

$y=\frac{57-9x}{7}=8-x-\frac{2x-1}{7}$; let $\frac{2x-1}{7}=m$

$x=\frac{7m+1}{2}=3m+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=6n-3+n=7n-3$

$y=8-7n+3-2n+1=12-9n$

when $n=1$, $x=4$, $y=3$.

14. $\frac{x}{11}-\frac{y}{13}=\frac{82}{143}$; $13x-11y=82$

$y=\frac{13x-82}{11}=x-7+\frac{2x-5}{11}$; let $\frac{2x-5}{11}=m$

$x=\frac{11m+5}{2}=5m+2+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=10n-5+2+n=11n-3$

$y=11n-3-7+2n-1=13n-11$

when $n=1$, $x=8$, $y=2$.

15. Let x be the number of florins, y the number of half-crowns; then

$4x+5y=58$; $x=14-y-\frac{y-2}{4}$; let $\frac{y-2}{4}=m$

Then $y=4m+2$ and $x=12-5m$, etc.

16. Let x be the number of half-guineas, y the number of half-crowns; then

$21x+5y=800$; $y=160-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$ and $y=160-21m$

when $m=1$, $x=5$, $y=139$, etc.

17. Let x be the number; y the first quotient, z the second quotient.

Then $\frac{x}{5}=y+\frac{2}{5}$; $\frac{x}{9}=z+\frac{3}{9}$

Hence $5y+2=9z+3$; $y=z+\frac{4z+1}{5}$; let $\frac{4z+1}{5}=m$

$z=\frac{5m-1}{4}=m+\frac{m-1}{4}$; let $\frac{m-1}{4}=n$; $m=4n+1$

$z=5n+1$; $y=9n+2$

Hence $y=2, 11 \ldots$ and hence $x=12, 57 \ldots$

18. Let x be the number of guineas, and y the number of crowns.

Then $21x+5y=235$; $y=47-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$; and $y=47-21m$, etc.

19. Let x be the number of half-guineas, and y the number of half-crowns.

Then $21x+5y=183$; $y=36-4x-\frac{x-3}{5}$ let $\frac{x-3}{5}=m$

Then $x=5m+3$; and $y=36-20m-12-m=24-21m$, etc.

20. Divide by 17; then $19x-31y=\frac{1000}{17}$; and the right-hand side being a fraction, no integral values of x and y can make the left side equal to the right.

21. Let x be the number of oxen, y of sheep, z of hens.
Then $x+y+z=100$, and $100x+20y+z=2000$,
$\therefore 99x+19y=1900$
Hence $y=100-5x-\frac{4x}{19}$; let $\frac{4x}{19}=m$
$x=\frac{19m}{4}=5m-\frac{m}{4}$; let $\frac{m}{4}=n$; $m=4n$
Then $x=19n$; $y=100-99n$
If $n=1$, $x=19$, $y=1$, and $\therefore z=80$.

22. Let A give x sixpences and receive y fourpenny pieces.
Then $6x-4y=58$; or $3x-2y=29$
Then $y=x-14+\frac{x-1}{2}$; let $\frac{x-1}{2}=m$
Then $x=2m+1$ and $y=3m-13$, etc.

23. Let x be the number of half-crowns, y of florins, z of shillings.
Then $x+y=4z$, and $\frac{5x}{2}+2y+z=244$
Hence $x+y=4z$, and $10x+8y+4z=976$
$\therefore 11x+9y=976$
Hence $y=108-x-\frac{2x-4}{9}$; let $\frac{2x-4}{9}=m$
$x=4m+2+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$
Then $x=9n+2$; $y=108-9n-2-2n=106-11n$
Hence $x=2$, $y=106$, $z=27$, which gives the greatest number of coins possible.

24. Let there be x half-crowns, y florins, and z fourpenny-pieces.

Then $x+y+z=50$, and $30x+24y+4z=1200$.

Hence $26x+20y=1000$.

$\therefore y=50-x-\frac{3x}{10}$; let $\frac{3x}{10}=m$.

Then $x=\frac{10m}{3}=3m+\frac{m}{3}$; let $\frac{m}{3}=n$; $m=3n$.

Hence $x=10n$ and $y=50-13n$

$\therefore x=10, 20, 30$; $y=37, 24, 11$; $z=3, 6, 9$.

25. Let A give x sovereigns, and receive y dollars.

Then $240x-51y=12$; or, $80x-17y=4$.

Hence $y=4x+\frac{12x-4}{17}$; let $\frac{12x-4}{17}=m$

$x=m+\frac{5m+4}{12}$; let $\frac{5m+4}{12}=n$

$m=2n+\frac{2n-4}{5}$; let $\frac{2n-4}{5}=p$

$n=2p+2+\frac{p}{2}$; let $\frac{p}{2}=q$; $p=2q$.

Then $x=m+n=3n+p=6p+6+5q=17q+6$

$y=4x+m=68q+24+2n+p=82q+28$.

Hence when $q=0$, $x=6$, $y=28$.

26. Let $2x$ and $3y$ be the parts; then $2x+3y=25$.

Then $x=12-y-\frac{y-1}{2}$; let $\frac{y-1}{2}=m$

$y=2m+1$ and $x=11-3m$

When $m=0$, $y=1$, $x=11$; $2x=22$, $3y=3$, etc.

27. Let there be x crowns and y florins; then $5x+2y=49$.

Then $y=24-2x-\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

$x=2m+1$, and $y=22-5m$

Hence we get positive values for x and y, when $m=0, 1, 2, 3, 4$.

28. Let $7x$ and $11y$ be the parts; then $7x+11y=100$.

Then $x=14-y-\frac{4y-2}{7}$; let $\frac{4y-2}{7}=m$.

$$y=\frac{7m+2}{4}=m+\frac{3m+2}{4}\text{; let }\frac{3m+2}{4}=n$$

$$m=\frac{4n-2}{3}=n+\frac{n-2}{3}\text{; let }\frac{n-2}{3}=p\text{; }n=3p+2.$$

Hence $y=2n+p=7p+4$

$x=14-7p-4-n-p=8-11p$.

When $p=0$, $y=4$, $x=8$; $7x=56$, $11y=44$.

29. Let the quotients be x and y.

Then $5x+2$ and $7y+4$ are the dividends

$\therefore\ 5x+7y+6=100$, or, $5x+7y=94$.

Then $x=18-y-\frac{2y-4}{5}$; let $\frac{2y-4}{5}=m$

$$y=2m+2+\frac{m}{2}\text{; let }\frac{m}{2}=n\text{; }m=2n.$$

Then $y=5n+2$ and $x=16-7n$.

Hence $x=16, 9, 2$; and $y=2, 7, 12$

and $5x+2=82, 47, 12$; and $7y+4=18, 53, 88$.

30. The L. C. M. of 2, 3, 4, 5, 6 is 60.

Hence the general expression for the number required is $60m+1$.

Making $m=1, 2, 3$ . . . . successively, we get 61, 121, 181, 241, 301, 361, and of these 301 is the only multiple of 7.

CI.

1.
$$\begin{array}{lllll}
x^{2p}+x^py^p+y^{2p} & & & & \\
x^{2p}-x^py^p+y^{2p} & & & & \\
\hline
x^{4p} & +x^{3p}y^p & +x^{2p}y^{2p} & & \\
 & -x^{3p}y^p & -x^{2p}y^{2p} & -x^py^{3p} & \\
 & & +x^{2p}y^{2p} & +x^py^{3p} & +y^{4p} \\
\hline
x^{4p} & & +x^{2p}y^{2p} & & +y^{4p}
\end{array}$$

2.

$$\begin{array}{l}
a^{3m}+3a^{2m}y^{n}+9a^{m}y^{2n}+27y^{3n}\\
a^{m}-3y^{n}\\
\hline
a^{4m}+3a^{3m}y^{n}+9a^{2m}y^{2n}+27a^{m}y^{3n}\\
\qquad -3a^{3m}y^{n}-9a^{2m}y^{2n}-27a^{m}y^{3n}-81y^{4n}\\
\hline
a^{4m} \qquad\qquad\qquad\qquad\qquad\qquad\qquad -81y^{4n}
\end{array}$$

3.

$$\begin{array}{l}
x^{4d}-2ax^{2d}+4a^{2}\\
x^{4d}+2ax^{2d}+4a^{2}\\
\hline
x^{8d}-2ax^{6d}+4a^{2}x^{4d}\\
\qquad +2ax^{6d}-4a^{2}x^{4d}+8a^{3}x^{2d}\\
\qquad\qquad\qquad +4a^{2}x^{4d}-8a^{3}x^{2d}+16a^{4}\\
\hline
x^{8d}\qquad\qquad +4a^{2}x^{4d}\qquad\qquad +16a^{4}
\end{array}$$

4.

$$\begin{array}{l}
a^{m}+b^{n}+c^{r}\\
a^{m}-b^{n}+c^{r}\\
\hline
a^{2m}+a^{m}b^{n}+a^{m}c^{r}\\
\qquad -a^{m}b^{n}-b^{2n}-b^{n}c^{r}\\
\qquad\qquad\qquad +a^{m}c^{r}+b^{n}c^{r}+c^{2r}\\
\hline
a^{2m}-b^{2n}+c^{2r}+2a^{m}c^{r}
\end{array}$$

5.

$$\begin{array}{l}
a^{m}+b^{n}-2c^{r}\\
2a^{m}-b+c^{2}\\
\hline
2a^{2m}+2a^{m}b^{n}-4a^{m}c^{r}\\
\qquad -a^{m}b-b^{n+1}+2bc^{r}\\
\qquad\qquad +a^{m}c^{2}+b^{n}c^{2}-2c^{r+2}\\
\hline
2a^{2m}+2a^{m}b^{n}-4a^{m}c^{r}-a^{m}b-b^{n+1}+2bc^{r}+a^{m}c^{2}+b^{n}c^{2}-2c^{r+2}
\end{array}$$

6.

$$\begin{array}{l}
x^{mn-n}-y^{m}\\
x^{n}+y^{mn-n}\\
\hline
x^{mn}-x^{n}y^{m}\\
\qquad +x^{mn-n}.y^{mn-n}-y^{mn-n+m}\\
\hline
x^{mn}+x^{mn-n}.y^{mn-n}-x^{n}y^{m}-y^{mn-n+m}
\end{array}$$

7.
$$\begin{array}{l}
x^{2n} - x^n y^n + y^{2n} \\
x^{2n} + x^n y^n + y^{2n} \\
\hline
x^{4n} - x^{3n}y^n + x^{2n}y^{2n} \\
\qquad + x^{3n}y^n - x^{2n}y^{2n} + x^n y^{3n} \\
\qquad\qquad\qquad + x^{2n}y^{2n} - x^n y^{3n} + y^{4n} \\
\hline
x^{4n} \qquad\qquad + x^{2n}y^{2n} \qquad\qquad + y^{4n}
\end{array}$$

8.
$$\begin{array}{l}
a^{p^2+p} - b^{p^2} + c^p \\
a^{p^2-p} + b^{1-p^2} + c^{1-p} \\
\hline
a^{2p^2} - a^{p^2-p}b^{p^2} + a^{p^2-p}c^p \\
\qquad + a^{p^2+p}b^{1-p^2} - b + b^{1-p^2}c^p \\
\qquad\qquad\qquad + a^{p^2+p}c^{1-p} - b^{p^2}c^{1-p} + c \\
\hline
a^{2p^2} - a^{p^2-p}b^{p^2} + a^{p^2-p}c^p + a^{p^2+p}b^{1-p^2} - b + b^{1-p^2}c^p \\
\qquad\qquad\qquad\qquad\qquad\qquad + a^{p^2+p}c^{1-p} - b^{p^2}c^{1-p} + c
\end{array}$$

9.
$$\begin{array}{l}
x^{2p} + x^p + 1 \\
x^{2p} + x^p + 1 \\
\hline
x^{4p} + x^{3p} + x^{2p} \\
\qquad + x^{3p} + x^{2p} + x^p \\
\qquad\qquad + x^{2p} + x^p + 1 \\
\hline
x^{4p} + 2x^{3p} + 3x^{2p} + 2x^p + 1
\end{array}$$

10.
$$\begin{array}{l}
x^{2p} - x^p + 1 \\
x^{2p} - x^p + 1 \\
\hline
x^{4p} - x^{3p} + x^{2p} \\
\qquad - x^{3p} + x^{2p} - x^p \\
\qquad\qquad + x^{2p} - x^p + 1 \\
\hline
x^{4p} - 2x^{3p} + 3x^{2p} - 2x^p + 1
\end{array}$$

CII.

1.
$$\begin{array}{l}
x^m - y^m)\,x^{4m} - y^{4m}\,(x^{3m} + x^{2m}y^m + x^m y^{2m} + y^{3m} \\
\qquad\quad \underline{x^{4m} - x^{3m}y^m} \\
\qquad\qquad\quad x^{3m}y^m - y^{4m} \\
\qquad\qquad\quad \underline{x^{3m}y^m - x^{2m}y^{2m}} \\
\qquad\qquad\qquad\quad x^{2m}y^{2m} - y^{4m} \\
\qquad\qquad\qquad\quad \underline{x^{2m}y^{2m} - x^m y^{3m}} \\
\qquad\qquad\qquad\qquad\quad x^m y^{3m} - y^{4m} \\
\qquad\qquad\qquad\qquad\quad \underline{x^m y^{3m} - y^{4m}}
\end{array}$$

2.
$$\begin{array}{l}
x^n + y^n)\, x^{5n} + y^{5n}\,(x^{4n} - x^{3n}y^n + x^{2n}y^{2n} - x^ny^{3n} + y^{4n} \\
\underline{x^{5n} + x^{4n}y^n} \\
-x^{4n}y^n + y^{5n} \\
\underline{-x^{4n}y^n - x^{3n}y^{2n}} \\
x^{3n}y^{2n} + y^{5n} \\
\underline{x^{3n}y^{2n} + x^{2n}y^{3n}} \\
-x^{2n}y^{3n} + y^{5n} \\
\underline{-x^{2n}y^{3n} - x^ny^{4n}} \\
x^ny^{4n} + y^{5n} \\
\underline{x^ny^{4n} + y^{5n}}
\end{array}$$

3.
$$\begin{array}{l}
x^r - y^r)\, x^{6r} - y^{6r}\,(x^{5r} + x^{4r}y^r, \text{ etc.} \\
\underline{x^{6r} - x^{5r}y^r} \\
x^{5r}y^r - y^{6r}, \text{ etc.}
\end{array}$$

4.
$$\begin{array}{l}
a^{3p} + b^{2q})\, a^{15p} + b^{10q}\,(a^{12p} - a^{9p}b^{2q}, \text{ etc.} \\
\underline{a^{15p} + a^{12p}b^{2q}} \\
-a^{12p}b^{2q} + b^{10q}, \text{ etc.}
\end{array}$$

5.
$$\begin{array}{l}
x^d - 3)\, x^{5d} - 243)\, x^{4d} + 3x^{3d}, \text{ etc.} \\
\underline{x^{5d} - 3x^{4d}} \\
3x^{4d} - 243, \text{ etc.}
\end{array}$$

6.
$$\begin{array}{l}
a^{2m} + 2a^mx^n + 4x^{2n})\, a^{4m} + 4a^{2m}x^{2n} + 16x^{4n}\,(a^{2m} - 2a^mx^n + 4x^{2n} \\
\underline{a^{4m} + 2a^{3m}x^n + 4a^{2m}x^{2n}} \\
-2a^{3m}x^n + 16x^{4n} \\
\underline{-2a^{3m}x^n - 4a^{2m}x^{2n} - 8a^mx^{3n}} \\
4a^{2m}x^{2n} + 8a^mx^{3n} + 16x^{4n} \\
\underline{4a^{2m}x^{2n} + 8a^mx^{3n} + 16x^{4n}}
\end{array}$$

7. $1+5x^p+x^{2p})\,2+9x^p+14x^{3p}+3x^{4p}\,(2-x^p+3x^{2p}$

$2+10x^p+2x^{2p}$

$-x^p-2x^{2p}+14x^{3p}$

$-x^p-5x^{2p}-x^{3p}$

$3x^{2p}+15x^{3p}+3x^{4p}$

$3x^{2p}+15x^{3p}+3x^{4p}$

8. $b^mc^{2m}-2b^{2m}c^m+b^{3m})\,4b^{2m}c^{3m}-13b^{3m}c^{2m}+14b^{4m}c^m-5b^{5m}\,(4b^mc^m-5b^{2m}$

$4b^{2m}c^{3m}-8b^{3m}c^{2m}+4b^{4m}c^m$

$-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}$

$-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}$

9. $a^{6m}+6a^{5m}+15a^{4m}+20a^{3m}+15a^{2m}+6a^m+1\,(a^{3m}+3a^{2m}+3a^m+1$

a^{6m}

$2a^{3m}+3a^{2m}$ \ $6a^{5m}+15a^{4m}$

$6a^{5m}+9a^{4m}$

$2a^{3m}+6a^{2m}+3a^m$ \ $6a^{4m}+20a^{3m}+15a^{2m}$

$6a^{4m}+18a^{3m}+9a^{2m}$

$2a^{3m}+6a^{2m}+6a^m+1$ \ $2a^{3m}+6a^{2m}+6a^m+1$

$2a^{3m}+6a^{2m}+6a^m+1$

10. $a^{2m}+2a^mb^n+b^{2n}+2a^mc^r+2b^nc^r+c^{2r}\,(a^m+b^n+c^r$

a^{2m}

$2a^m+b^n$ \ $2a^mb^n+b^{2n}$

$2a^mb^n+b^{2n}$

$2a^m+2b^n+c^r$ \ $2a^mc^r+2b^nc^r+c^{2r}$

$2a^mc^r+2b^nc^r+c^{2r}$

CIII.

1.
$$\begin{array}{l} x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1 \\ x^{\frac{1}{3}} - 1 \\ \hline x - 2x^{\frac{2}{3}} + x^{\frac{1}{3}} \\ \quad - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 1 \\ \hline x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1 \end{array}$$

2.
$$\begin{array}{l} y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1 \\ y^{\frac{1}{4}} - 1 \\ \hline y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} \\ \quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1 \\ \hline y \qquad\qquad\qquad - 1 \end{array}$$

3.
$$\begin{array}{l} a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}} \\ a^{\frac{2}{3}} - x^{\frac{2}{3}} \\ \hline a^2 + a^{\frac{4}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{4}{3}} \\ \quad - a^{\frac{4}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2 \\ \hline a^2 \qquad\qquad\qquad - x^2 \end{array}$$

4.
$$\begin{array}{l} a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}} \\ a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \\ \hline a - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}c^{\frac{2}{3}} \\ \quad + a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{2}{3}}c^{\frac{1}{3}} + b^{\frac{1}{3}}c^{\frac{2}{3}} \\ \qquad + a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{2}{3}} + c \\ \hline a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} \end{array}$$

5.
$$\begin{array}{l} 5x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y^{\frac{1}{4}} + 3x^{\frac{1}{4}}y^{\frac{1}{2}} + 7y^{\frac{3}{4}} \\ 2x^{\frac{1}{4}} - 3y^{\frac{1}{4}} \\ \hline 10x + 4x^{\frac{3}{4}}y^{\frac{1}{4}} + 6x^{\frac{1}{2}}y^{\frac{1}{2}} + 14x^{\frac{1}{4}}y^{\frac{3}{4}} \\ \quad - 15x^{\frac{3}{4}}y^{\frac{1}{4}} - 6x^{\frac{1}{2}}y^{\frac{1}{2}} - 9x^{\frac{1}{4}}y^{\frac{3}{4}} - 21y \\ \hline 10x - 11x^{\frac{3}{4}}y^{\frac{1}{4}} \qquad + 5x^{\frac{1}{4}}y^{\frac{3}{4}} - 21y \end{array}$$

6.
$$\begin{array}{l} m^{\frac{4}{5}} + m^{\frac{3}{5}}n^{\frac{1}{5}} + m^{\frac{2}{5}}n^{\frac{2}{5}} + m^{\frac{1}{5}}n^{\frac{3}{5}} + n^{\frac{4}{5}} \\ m^{\frac{1}{5}} - n^{\frac{1}{5}} \\ \hline m + m^{\frac{4}{5}}n^{\frac{1}{5}} + m^{\frac{3}{5}}n^{\frac{2}{5}} + m^{\frac{2}{5}}n^{\frac{3}{5}} + m^{\frac{1}{5}}n^{\frac{4}{5}} \\ \quad - m^{\frac{4}{5}}n^{\frac{1}{5}} - m^{\frac{3}{5}}n^{\frac{2}{5}} - m^{\frac{2}{5}}n^{\frac{3}{5}} - m^{\frac{1}{5}}n^{\frac{4}{5}} - n \\ \hline m \qquad\qquad\qquad\qquad - n \end{array}$$

7.
$$\begin{array}{l} m^{\frac{2}{3}} - 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\ m^{\frac{2}{3}} + 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\ \hline m^{\frac{4}{3}} - 2d^{\frac{1}{4}}m \quad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} \\ \qquad + 2d^{\frac{1}{4}}m \quad - 4d^{\frac{1}{2}}m^{\frac{2}{3}} + 8d^{\frac{3}{4}}m^{\frac{1}{3}} \\ \qquad\qquad\qquad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} - 8d^{\frac{3}{4}}m^{\frac{1}{3}} + 16d \\ \hline m^{\frac{4}{3}} \qquad\qquad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} \qquad\qquad + 16d \end{array}$$

8.
$$\begin{array}{l} 8a^{\frac{6}{7}} + 4a^{\frac{4}{7}}b^{\frac{2}{7}} + 5a^{\frac{2}{7}}b^{\frac{4}{7}} + 9b^{\frac{6}{7}} \\ 2a^{\frac{1}{7}} - 3b^{\frac{1}{7}} \\ \hline 16a + 8a^{\frac{5}{7}}b^{\frac{2}{7}} + 10a^{\frac{3}{7}}b^{\frac{4}{7}} + 18a^{\frac{1}{7}}b^{\frac{6}{7}} \\ \qquad - 24a^{\frac{6}{7}}b^{\frac{1}{7}} - 12a^{\frac{4}{7}}b^{\frac{3}{7}} - 15a^{\frac{2}{7}}b^{\frac{5}{7}} - 27b \\ \hline 16a + 8a^{\frac{5}{7}}b^{\frac{2}{7}} + 10a^{\frac{3}{7}}b^{\frac{4}{7}} + 18a^{\frac{1}{7}}b^{\frac{6}{7}} - 24a^{\frac{6}{7}}b^{\frac{1}{7}}, \text{ etc.} \end{array}$$

9.
$$\begin{array}{l} x^{\frac{1}{3}} + a^{\frac{1}{3}} \\ x^{\frac{1}{3}} + a^{\frac{1}{3}} \\ \hline x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} \\ \quad + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\ \hline x^{\frac{2}{3}} + 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \end{array}$$

10.
$$\begin{array}{l} x^{\frac{1}{3}} - a^{\frac{1}{3}} \\ x^{\frac{1}{3}} - a^{\frac{1}{3}} \\ \hline x^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} \\ \quad - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\ \hline x^{\frac{2}{3}} - 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \end{array}$$

11.
$$\begin{array}{l} x^{\frac{2}{3}} + y^{\frac{2}{3}} \\ x^{\frac{2}{3}} + y^{\frac{2}{3}} \\ \hline x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} \\ \quad + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} \\ \hline x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} \end{array}$$

12.
$$\begin{array}{l} a + b^{\frac{1}{4}} \\ a + b^{\frac{1}{4}} \\ \hline a^2 + ab^{\frac{1}{4}} \\ \quad + ab^{\frac{1}{4}} + b^{\frac{1}{2}} \\ \hline a^2 + 2ab^{\frac{1}{4}} + b^{\frac{1}{2}} \end{array}$$

13.
$$\begin{array}{l} x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\ x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\ \hline x - 2x^{\frac{3}{4}} + 3x^{\frac{1}{2}} \\ \quad - 2x^{\frac{3}{4}} + 4x^{\frac{1}{2}} - 6x^{\frac{1}{4}} \\ \qquad\qquad + 3x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + 9 \\ \hline x - 4x^{\frac{3}{4}} + 10x^{\frac{1}{2}} - 12x^{\frac{1}{4}} + 9 \end{array}$$

14.
$$\begin{array}{l} 2x^{\frac{2}{7}} + 3x^{\frac{1}{7}} + 4 \\ 2x^{\frac{2}{7}} + 3x^{\frac{1}{7}} + 4 \\ \hline 4x^{\frac{4}{7}} + 6x^{\frac{3}{7}} + 8x^{\frac{2}{7}} \\ \qquad + 6x^{\frac{3}{7}} + 9x^{\frac{2}{7}} + 12x^{\frac{1}{7}} \\ \qquad\qquad + 8x^{\frac{2}{7}} + 12x^{\frac{1}{7}} + 16 \\ \hline 4x^{\frac{4}{7}} + 12x^{\frac{3}{7}} + 25x^{\frac{2}{7}} + 24x^{\frac{1}{7}} + 16 \end{array}$$

15.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}} \\
\underline{x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}} \\
x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{1}{3}} \\
\quad - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} \\
\underline{\qquad\quad + x^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}} \\
x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}}z^{\frac{1}{3}} - 2y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}
\end{array}$$

16.
$$\begin{array}{l}
x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}} \\
\underline{x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}}} \\
x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} - x^{\frac{1}{4}}z^{\frac{1}{4}} \\
\quad + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} \\
\underline{\qquad\quad - x^{\frac{1}{4}}z^{\frac{1}{4}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}} \\
x^{\frac{1}{2}} + 4x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2x^{\frac{1}{4}}z^{\frac{1}{4}} - 4y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}
\end{array}$$

CIV.

1.
$$\begin{array}{l}
x^{\frac{1}{2}} - y^{\frac{1}{2}})\, x - y\, (x^{\frac{1}{2}} + y^{\frac{1}{2}} \\
\qquad\qquad \underline{x - x^{\frac{1}{2}}y^{\frac{1}{2}}} \\
\qquad\qquad\quad x^{\frac{1}{2}}y^{\frac{1}{2}} - y \\
\qquad\qquad\quad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - y}
\end{array}$$

2.
$$\begin{array}{l}
a^{\frac{1}{2}} + b^{\frac{1}{2}})\, a - b\, (a^{\frac{1}{2}} - b^{\frac{1}{2}} \\
\qquad\qquad \underline{a + a^{\frac{1}{2}}b^{\frac{1}{2}}} \\
\qquad\qquad\quad - a^{\frac{1}{2}}b^{\frac{1}{2}} - b \\
\qquad\qquad\quad \underline{- a^{\frac{1}{2}}b^{\frac{1}{2}} - b}
\end{array}$$

3.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}})\, x - y\, (x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\qquad\qquad \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} \\
\qquad\qquad\quad x^{\frac{2}{3}}y^{\frac{1}{3}} - y \\
\qquad\qquad\quad \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}}} \\
\qquad\qquad\qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
\qquad\qquad\qquad\qquad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

4.
$$\begin{array}{l}
a^{\frac{1}{3}} + b^{\frac{1}{3}})\, a + b\, (a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\qquad\qquad \underline{a + a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
\qquad\qquad\quad - a^{\frac{2}{3}}b^{\frac{1}{3}} + b \\
\qquad\qquad\quad \underline{- a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}} \\
\qquad\qquad\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}} + b \\
\qquad\qquad\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}} + b}
\end{array}$$

5. $x^{\frac{1}{5}}+y^{\frac{1}{5}})\,x+y\,(x^{\frac{4}{5}}-x^{\frac{3}{5}}y^{\frac{1}{5}}$, etc.

$\underline{x+x^{\frac{4}{5}}y^{\frac{1}{5}}}$

$-x^{\frac{4}{5}}y^{\frac{1}{5}}+y$, etc.

6. $m^{\frac{1}{5}}-n^{\frac{1}{5}})\,m-n\,(m^{\frac{4}{5}}+m^{\frac{3}{5}}n^{\frac{1}{5}}$, etc.

$\underline{m-m^{\frac{4}{5}}n^{\frac{1}{5}}}$

$m^{\frac{4}{5}}n^{\frac{1}{5}}-n$, etc.

7. $x^{\frac{1}{4}}-3y^{\frac{1}{4}})\,x-81y\,(x^{\frac{3}{4}}+3x^{\frac{1}{2}}y^{\frac{1}{4}}$, etc.

$\underline{x-3x^{\frac{3}{4}}y^{\frac{1}{4}}}$

$3x^{\frac{3}{4}}y^{\frac{1}{4}}-81y$, etc.

8. $3a^{\frac{1}{4}}-2b^{\frac{1}{4}})\,81a-16b\,(27a^{\frac{3}{4}}+18a^{\frac{1}{2}}b^{\frac{1}{4}}$, etc.

$\underline{81a-54a^{\frac{3}{4}}b^{\frac{1}{4}}}$

$54a^{\frac{3}{4}}b^{\frac{1}{4}}-16b$, etc.

9. $a^{\frac{1}{2}}+x^{\frac{1}{2}})\,a-x\,(a^{\frac{1}{2}}-x^{\frac{1}{2}}$

$\underline{a+a^{\frac{1}{2}}x^{\frac{1}{2}}}$

$-a^{\frac{1}{2}}x^{\frac{1}{2}}-x$

$\underline{-a^{\frac{1}{2}}x^{\frac{1}{2}}-x}$

10. $m^{\frac{1}{5}}-3)\,m-243\,(m^{\frac{4}{5}}+3m^{\frac{3}{5}}$, etc.

$\underline{m-3m^{\frac{4}{5}}}$

$3m^{\frac{4}{5}}-243$

$\underline{3m^{\frac{4}{5}}-9m^{\frac{3}{5}}}$, etc.

11. $x^{\frac{1}{2}}+7)\,x+17x^{\frac{1}{2}}+70\,(x^{\frac{1}{2}}+10$

$\underline{x+7x^{\frac{1}{2}}}$

$10x^{\frac{1}{2}}+70$

$\underline{10x^{\frac{1}{2}}+70}$

12. $x^{\frac{1}{3}}-3)\,x^{\frac{2}{3}}+x^{\frac{1}{3}}-12\,(x^{\frac{1}{3}}+4$

$\underline{x^{\frac{2}{3}}-3x^{\frac{1}{3}}}$

$4x^{\frac{1}{3}}-12$

$\underline{4x^{\frac{1}{3}}-12}$

13. $b^{\frac{1}{4}}-1)\,-b^{\frac{5}{4}}+3b-3b^{\frac{3}{4}}+b^{\frac{1}{2}}\,(-b+2b^{\frac{3}{4}}-b^{\frac{1}{2}}$

$\underline{-b^{\frac{5}{4}}+b}$

$2b-3b^{\frac{3}{4}}$

$\underline{2b-2b^{\frac{3}{4}}}$

$-b^{\frac{3}{4}}+b^{\frac{1}{2}}$

$\underline{-b^{\frac{3}{4}}+b^{\frac{1}{2}}}$

14. $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})\,x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$

$$\begin{array}{l}
\underline{x+x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{2}{3}}z^{\frac{1}{3}}}\\
-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{2}{3}}z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+y+z\\
\underline{-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}}\\
x^{\frac{1}{3}}y^{\frac{2}{3}}+y-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+z\\
\underline{x^{\frac{1}{3}}y^{\frac{2}{3}}+y+y^{\frac{2}{3}}z^{\frac{1}{3}}}\\
-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+z\\
\underline{-x^{\frac{2}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}z^{\frac{2}{3}}}\\
-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{2}{3}}+z\\
\underline{-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{2}{3}}}\\
x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z\\
\underline{x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z}
\end{array}$$

15. $x^{\frac{1}{3}}+4)\,x-5x^{\frac{2}{3}}-46x^{\frac{1}{3}}-40\,(x^{\frac{2}{3}}-9x^{\frac{1}{3}}-10$

$$\begin{array}{l}
\underline{x+4x^{\frac{2}{3}}}\\
-9x^{\frac{2}{3}}-46x^{\frac{1}{3}}\\
\underline{-9x^{\frac{2}{3}}-36x^{\frac{1}{3}}}\\
-10x^{\frac{1}{3}}-40\\
\underline{-10x^{\frac{1}{3}}-40}
\end{array}$$

16. $m^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}})\,m+m^{\frac{1}{2}}n^{\frac{1}{2}}+n\,(m^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}}$

$$\begin{array}{l}
\underline{m-m^{\frac{3}{4}}n^{\frac{1}{4}}+m^{\frac{1}{2}}n^{\frac{1}{2}}}\\
m^{\frac{3}{4}}n^{\frac{1}{4}}+n\\
\underline{m^{\frac{3}{4}}n^{\frac{1}{4}}-m^{\frac{1}{2}}n^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{3}{4}}}\\
m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n\\
\underline{m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n}
\end{array}$$

17. $p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1)\,p - 4p^{\frac{3}{4}} + 6p^{\frac{1}{2}} - 4p^{\frac{1}{4}} + 1\,(p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1$

$$\begin{array}{l}
p - 2p^{\frac{3}{4}} + p^{\frac{1}{2}} \\ \hline
-2p^{\frac{3}{4}} + 5p^{\frac{1}{2}} - 4p^{\frac{1}{4}} \\
-2p^{\frac{3}{4}} + 4p^{\frac{1}{2}} - 2p^{\frac{1}{4}} \\ \hline
p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1 \\
p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1 \\ \hline
\end{array}$$

18. $2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}})\,2x + x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - x^{\frac{1}{2}}z^{\frac{1}{2}} - z\,(x^{\frac{1}{2}} - y^{\frac{1}{2}} - z^{\frac{1}{2}}$

$$\begin{array}{l}
2x + 3x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}z^{\frac{1}{2}} \\ \hline
-2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - 2x^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
-2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - y^{\frac{1}{2}}z^{\frac{1}{2}} \\ \hline
-2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
-2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z \\ \hline
\end{array}$$

19. $x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}})\,x + y\,(x^{\frac{1}{5}} + y^{\frac{1}{5}}$

$$\begin{array}{l}
x - x^{\frac{4}{5}}y^{\frac{1}{5}} + x^{\frac{3}{5}}y^{\frac{2}{5}} - x^{\frac{2}{5}}y^{\frac{3}{5}} + x^{\frac{1}{5}}y^{\frac{4}{5}} \\ \hline
x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y \\
x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y \\ \hline
\end{array}$$

CV.

1.
$$\begin{array}{ll}
a^{-1} + b^{-1} & \\
a^{-1} - b^{-1} & \\ \hline
a^{-2} + a^{-1}b^{-1} & \\
\quad\; - a^{-1}b^{-1} & - b^{-2} \\ \hline
a^{-2} & - b^{-2}
\end{array}$$

2.
$$\begin{array}{ll}
x^{-3} + b^{-2} & \\
x^{-3} - b^{-2} & \\ \hline
x^{-6} + x^{-3}b^{-2} & \\
\quad\; - x^{-3}b^{-2} & - b^{-4} \\ \hline
x^{-6} & - b^{-4}
\end{array}$$

3.
$$\begin{array}{l}
x^3+x+x^{-1}+x^{-3}\\
x-x^{-1}\\
\hline
x^4+x^2+1+x^{-2}\\
\qquad -x^2-1-x^{-2}-x^{-4}\\
\hline
x^4 \qquad\qquad\qquad -x^{-4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+x^{-2}\\
x^2+1+x^{-2}\\
\hline
x^4-x^2+1\\
\quad +x^2-1+x^{-2}\\
\qquad\quad +1-x^{-2}+x^{-4}\\
\hline
x^4 \qquad +1+x^{-4}
\end{array}$$

5.
$$\begin{array}{l}
a^{-2}+b^{-2}\\
a^{-2}-b^{-2}\\
\hline
a^{-4}+a^{-2}b^{-2}\\
\qquad -a^{-2}b^{-1}-b^{-4}\\
\hline
a^{-4} \qquad\qquad -b^{-4}
\end{array}$$

6.
$$\begin{array}{l}
a^{-1}-b^{-1}+c^{-1}\\
a^{-1}+b^{-1}+c^{-1}\\
\hline
a^{-2}-a^{-1}b^{-1}+a^{-1}c^{-1}\\
\qquad +a^{-1}b^{-1}-b^{-2}+b^{-1}c^{-1}\\
\qquad\qquad\qquad +a^{-1}c^{-1}-b^{-1}c^{-1}+c^{-2}\\
\hline
a^{-2}+2a^{-1}c^{-1}-b^{-2}+c^{-2}
\end{array}$$

7.
$$\begin{array}{l}
1+ab^{-1}+a^2b^{-2}\\
1-ab^{-1}+a^2b^{-2}\\
\hline
1+ab^{-1}+a^2b^{-2}\\
\quad -ab^{-1}-a^2b^{-2}-a^3b^{-3}\\
\qquad\qquad +a^2b^{-2}+a^3b^{-3}+a^4b^{-4}\\
\hline
1 \qquad\qquad +a^2b^{-2}+a^4b^{-4}
\end{array}$$

8.
$$\begin{array}{l}
a^2b^{-2}+2+a^{-2}b^2\\
a^2b^{-2}-2-a^{-2}b^2\\
\hline
a^4b^{-4}+2a^2b^{-2}+1\\
\qquad -2a^2b^{-2}-4-2a^{-2}b^2\\
\qquad\qquad\qquad -1-2a^{-2}b^2-a^{-4}b^4\\
\hline
a^4b^{-4}-4 \quad -a^{-4}b^4-4a^{-2}b^2
\end{array}$$

9.
$$\begin{array}{r}
4x^{-3}+3x^{-2}+2x^{-1}+1\phantom{+3x^{-3}+2x^{-2}+x^{-1}+1} \\
x^{-2}-x^{-1}+1\phantom{+2x^{-1}+1+3x^{-3}+2x^{-2}+x^{-1}+1} \\
\hline
4x^{-5}+3x^{-4}+2x^{-3}+x^{-2}\phantom{+2x^{-2}+x^{-1}+1} \\
-4x^{-4}-3x^{-3}-2x^{-2}-x^{-1} \\
+4x^{-3}+3x^{-2}+2x^{-1}+1 \\
\hline
4x^{-5}-x^{-4}+3x^{-3}+2x^{-2}+x^{-1}+1
\end{array}$$

10.
$$\begin{array}{l}
\frac{5}{2}x^{-2}+3x^{-1}-\frac{7}{3} \\
2x^{-2}-x^{-1}-\frac{1}{2} \\
\hline
5x^{-4}+6x^{-3}-\frac{14}{3}x^{-2} \\
\qquad -\frac{5}{2}x^{-3}-3x^{-2}+\frac{7}{3}x^{-1} \\
\qquad\qquad\qquad -\frac{5}{4}x^{-2}-\frac{3}{2}x^{-1}+\frac{7}{6} \\
\hline
5x^{-4}+\frac{7}{2}x^{-3}-\frac{107}{12}x^{-2}+\frac{5}{6}x^{-1}+\frac{7}{6}
\end{array}$$

CVI.

1.
$$\begin{array}{l}
x+x^{-1})\,x^2-x^{-2}\,(x-x^{-1} \\
\qquad\quad x^2+1 \\
\qquad\quad \overline{} \\
\qquad\qquad -1-x^{-2} \\
\qquad\qquad \underline{-1-x^{-2}}
\end{array}$$

2.
$$\begin{array}{l}
a-b^{-1})\,a^2-b^{-2}\,(a+b^{-1} \\
\qquad\quad \underline{a^2-ab^{-1}} \\
\qquad\qquad ab^{-1}-b^{-2} \\
\qquad\qquad \underline{ab^{-1}-b^{-2}}
\end{array}$$

3.
$$\begin{array}{l}
m+n^{-1})\,m^3+n^{-3}\,(m^2-mn^{-1}+n^{-2} \\
\qquad\quad \underline{m^3+m^2n^{-1}} \\
\qquad\qquad -m^2n^{-1}+n^{-3} \\
\qquad\qquad \underline{-m^2n^{-1}-mn^{-2}} \\
\qquad\qquad\qquad mn^{-2}+n^{-3} \\
\qquad\qquad\qquad \underline{mn^{-2}+n^{-3}}
\end{array}$$

4. $c-d^{-1})c^5-d^{-5}(c^4+c^3d^{-1}+c^2d^{-2}$, etc.

$\underline{c^5-c^4d^{-1}}$

$c^4d^{-1}-d^{-5}$

$\underline{c^4d^{-1}-c^3d^{-2}}$

$c^3d^{-2}-d^{-5}$

$\underline{c^3d^{-2}-c^2d^{-3}}$, etc.

5. $xy^{-1}+x^{-1}y)x^2y^{-2}+2+x^{-2}y^2(xy^{-1}+x^{-1}y$

$\underline{x^2y^{-2}+1}$

$1+x^{-2}y^2$

$\underline{1+x^{-2}y^2}$

6. $a^{-2}-a^{-1}b^{-1}+b^{-2})a^{-4}+a^{-2}b^{-2}+b^{-4}(a^{-2}+a^{-1}b^{-1}+b^{\ 2}$

$\underline{a^{-4}-a^{-3}b^{-1}+a^{-2}b^{-2}}$

$a^{-3}b^{-1}+b^{-4}$

$\underline{a^{-3}b^{-1}-a^{-2}b^{-2}+a^{-1}b^{-3}}$

$a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}$

$\underline{a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}}$

7. $xy^{-1}-x^{-1}y)x^3y^{-3}-3xy^{-1}+3x^{-1}y-x^{-3}y^3(x^2y^{-2}-2+x^{-2}y^2$

$\underline{x^3y^{-3}-\ xy^{-1}}$

$-2xy^{-1}+3x^{-1}y$

$\underline{-2xy^{-1}+2x^{-1}y}$

$x^{-1}y-x^{-3}y^3$

$\underline{x^{-1}y-x^{-3}y^3}$

8. $\frac{x^{-2}}{2}-x^{-1}+3\Big)\frac{3x^{-5}}{4}-4x^{-4}+\frac{77x^{-3}}{8}-\frac{43x^{-2}}{4}-\frac{33x^{-1}}{4}+27\Big(\frac{3x^{-3}}{2}-5x^{-2}+\frac{x^{-1}}{4}+9$

$$\underline{\frac{3x^{-5}}{4}-\frac{3x^{-4}}{2}+\frac{9x^{-3}}{2}}$$

$$-\frac{5x^{-4}}{2}+\frac{41x^{-3}}{8}-\frac{43x^{-2}}{4}$$

$$\underline{-\frac{5x^{-4}}{2}+5x^{-3}-15x^{-2}}$$

$$\frac{x^{-3}}{8}+\frac{17x^{-2}}{4}-\frac{33x^{-1}}{4}$$

$$\underline{\frac{x^{-3}}{8}-\frac{x^{-2}}{4}+\frac{3x^{-1}}{4}}$$

$$\frac{9x^{-2}}{2}-9x^{-1}+27$$

$$\underline{\frac{9x^{-2}}{2}-9x^{-1}+27}$$

9. $ab^{-1}+a^{-1}b)\,a^3b^{-3}+a^{-3}b^3\,(a^2b^{-2}-1+a^{-2}b^2$

$$\underline{a^3b^{-3}+ab^{-1}}$$

$$-ab^{-1}+a^{-3}b^3$$

$$\underline{-ab^{-1}-a^{-1}b}$$

$$a^{-1}b+a^{-3}b^3$$

$$\underline{a^{-1}b+a^{-3}b^3}$$

10. $a^{-1}+b^{-1}+c^{-1})a^{-3}+b^{-3}+c^{-3}-3a^{-1}b^{-1}c^{-1}(a^{-2}-a^{-1}b^{-1}-a^{-1}c^{-1}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +b^{-2}-b^{-1}c^{-1}+c^{-2}$

$$
\begin{array}{l}
\underline{a^{-3}+a^{-2}b^{-1}+a^{-2}c^{-1}}\\
\quad -a^{-2}b^{-1}-a^{-2}c^{-1}-3a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}\\
\quad \underline{-a^{-2}b^{-1}-a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}}\\
\qquad -a^{-2}c^{-1}+a^{-1}b^{-2}-2a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}\\
\qquad \underline{-a^{-2}c^{-1}-a^{-1}b^{-1}c^{-1}-a^{-1}c^{-2}}\\
\qquad\quad a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}+b^{-3}+c^{-3}\\
\qquad\quad \underline{a^{-1}b^{-2}+b^{-3}+b^{-2}c^{-1}}\\
\qquad\qquad -a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}-b^{-2}c^{-1}+c^{-3}\\
\qquad\qquad \underline{-a^{-1}b^{-1}c^{-1}-b^{-2}c^{-1}-b^{-1}c^{-2}}\\
\qquad\qquad\quad a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}\\
\qquad\qquad\quad \underline{a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}}
\end{array}
$$

CVII.

1. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y)x^{\frac{4}{3}}-4xy+4x^{\frac{2}{3}}y+4y^{2}(x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y$

$$
\begin{array}{l}
\underline{x^{\frac{4}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2x^{\frac{2}{3}}y}\\
\quad -2x^{\frac{1}{3}}y^{\frac{1}{2}}-4xy+2x^{\frac{2}{3}}y\\
\quad \underline{-2x^{\frac{1}{3}}y^{\frac{1}{2}}-4xy-4x^{\frac{1}{3}}y^{\frac{3}{2}}}\\
\qquad 2x^{\frac{2}{3}}y+4x^{\frac{1}{3}}y^{\frac{3}{2}}+4y^{2}\\
\qquad \underline{2x^{\frac{2}{3}}y+4x^{\frac{1}{3}}y^{\frac{3}{2}}+4y^{2}}
\end{array}
$$

2. $\{x^{15ab}\times x^{12}\}^{\frac{1}{3a-2}}=x^{\frac{15ab+12}{3a-2}}$.

3. $(x^{10b+18a})^{\frac{1}{3a-2}}=x^{\frac{10b+18a}{3a-2}}$.

4. $\left\{\frac{x^{2}+a^{2}-x^{2}+a^{2}}{x^{4}-a^{4}}-\frac{a(x-a-x-a)}{x^{4}-a^{4}}\right\}=\left(\frac{4a^{2}}{x^{4}-a^{4}}\right)^{\frac{1}{2}}$, etc.

5. $$\begin{array}{l}
\frac{7}{3}x^{-2}+4x^{-1}-\frac{2}{7} \\
3x^{-2}-2x^{-1}-\frac{1}{2} \\
\hline
7x^{-4}+12x^{-3}-\frac{6}{7}x^{-2} \\
\qquad -\frac{14}{3}x^{-3}-8x^{-2}+\frac{4}{7}x^{-1} \\
\qquad\qquad -\frac{7}{6}x^{-2}-2x^{-1}+\frac{1}{7} \\
\hline
7x^{-4}+\frac{22}{3}x^{-3}-\frac{421}{42}x^{-2}-\frac{10}{7}x^{-1}+\frac{1}{7}
\end{array}$$

6. $$\frac{x^{a+b+a-b+c-2a}}{x^{c-a}}=x^{c-c+a}=x^{a}.$$

7. $$\begin{array}{l}
x^{n}+y^{n})\,x^{2n}-y^{2n}\,(x^{n}-y^{n} \\
\qquad\qquad \underline{x^{2n}+x^{n}y^{n}} \\
\qquad\qquad\qquad -x^{n}y^{n}-y^{2n} \\
\qquad\qquad\qquad \underline{-x^{n}y^{n}-y^{2n}}
\end{array}$$

8. $$\begin{array}{l}
a^{\frac{3}{2}}+3ab^{\frac{1}{3}}+3a^{\frac{1}{2}}b^{\frac{2}{3}}+b^{\frac{3}{3}} \\
a^{\frac{1}{2}}-b^{\frac{1}{3}} \\
\hline
a^{2}+3a^{\frac{3}{2}}b^{\frac{1}{3}}+3ab^{\frac{2}{3}}+a^{\frac{1}{2}}b^{\frac{3}{3}} \\
\quad -\ a^{\frac{3}{2}}b^{\frac{1}{3}}-3ab^{\frac{2}{3}}-3a^{\frac{1}{2}}b^{\frac{3}{3}}-b^{\frac{4}{3}} \\
\hline
a^{2}+2a^{\frac{3}{2}}b^{\frac{1}{3}}\qquad\qquad -2a^{\frac{1}{2}}b^{\frac{3}{3}}-b^{\frac{4}{3}}
\end{array}$$

9. $$\begin{array}{l}
a^{\frac{1}{3}}-b^{\frac{1}{3}})\,a-b\,(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}},\text{ etc.} \\
\qquad\qquad \underline{a-a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
\qquad\qquad\qquad a^{\frac{2}{3}}b^{\frac{1}{3}}-b,\text{ etc.}
\end{array}$$

10. $(a^2)^m = a^{2+2+\cdots \ m \text{ terms}} = a^{2m} = a^m \times a^m = (a^m)^2.$

11. $a^{m^n} = a^{mn}$; $\therefore m^n = mn$; $m^{n-1} = n$; $m = n^{\frac{1}{n-1}}.$

12. $x^{a+b+c+a+b-c+a-b+c+b+c-a} = x^{2a+2b+2c}.$

13. $(x^p)^p \div (x^p)^{p-q} = x^{p2} \div x^{p2-pq} = x^{pq}.$

14. $4a^x \div \dfrac{1}{4a^x} = 4a^x \times 4a^x = 16a^{2x}.$

15. $[\{a^{mn}\}^p] \div [\{a^{mn}\}^{-p}] = a^{mnp} \div a^{-mnp} = a^{2mnp}.$

16. $2a^{2m} + 2a^m b^p - 4a^m c^n - 3a^m b - 3b^{p+1} + 6bc^n.$

17. $a^{m-n+n-m}.$ $b^{n-p+p-n}.$ $c = 1 \times 1 \times c = c.$

18. $\dfrac{a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}})}{(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})} = \dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$

19. $(x^{\frac{1}{3}} + x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - x^{\frac{1}{6}} + 1) = x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$
$(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1) = x^{\frac{4}{3}} + x^{\frac{2}{3}} + 1$

20.
$$\begin{array}{llll}
a^m - ba^{m-1}x + ca^{m-2}x^2 & & & \\
a^n + ba^{n-1}x - ca^{n-2}x^2 & & & \\
\hline
a^{m+n} - ba^{m+n-1}x + ca^{m+n-2}x^2 & & & \\
\quad + ba^{m+n-1}x - b^2a^{m+n-2}x^2 + bca^{m+n-3}x^3 & & & \\
\qquad\qquad - ca^{m+n-2}x^2 + bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4 & & & \\
\hline
a^{m+n} - b^2a^{m+n-2}x^2 + 2bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4 & & &
\end{array}$$

21.
$$\begin{array}{rl}
x^{pq-p} + y^{pq-q}) & x^{2pq-2p} - y^{2pq-2q} \,(x^{pq-p} - y^{pq-q} \\
 & x^{2pq-2p} + x^{pq-p}y^{pq-q} \\
\hline
 & \qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
 & \qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
\hline
\end{array}$$

22. $\{(a^m)^{\frac{m^2-1}{m}}\}^{\frac{1}{m+1}} = a^{\frac{m^2-1}{m+1}} = a^{m-1}.$

23. $$\begin{array}{l} x^{3r}+x^{2r}y^{p}+x^{r}y^{2p}+y^{3p} \\ x^{r}-y^{p} \\ \hline x^{4r}+x^{3r}y^{p}+x^{2r}y^{2p}+x^{r}y^{3p} \\ \qquad -x^{3r}y^{p}-x^{2r}y^{2p}-x^{r}y^{3p}-y^{4p} \\ \hline x^{4r} \qquad\qquad\qquad\qquad\qquad -y^{4p} \end{array}$$

24. $\sqrt[4]{625}=\sqrt{25}=5$; and $\frac{1}{12^2}=\frac{1}{144}$.

25. $$\begin{array}{l} x^{mn-n}-y^{mn-m} \\ x^{n}-y^{m} \\ \hline x^{mn}-x^{n}y^{mn-m}-x^{mn-n}y^{m}+y^{mn} \end{array}$$

26. $$\begin{array}{l} x^{\frac{3}{4}}+3x^{\frac{1}{2}}-1 \\ x^{\frac{1}{4}}-2x^{-\frac{1}{4}} \\ \hline x+3x^{\frac{3}{4}}-x^{\frac{1}{4}} \\ \quad -2x^{\frac{1}{2}}-6x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \\ \hline x+3x^{\frac{3}{4}}-2x^{\frac{1}{2}}-7x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \end{array}$$

CVIII.

1. $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$; $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; $\sqrt[6]{x^3}$ and $\sqrt[6]{y^2}$.

2. $4^{\frac{1}{3}}$ and $2^{\frac{1}{5}}$; $4^{\frac{5}{15}}$ and $2^{\frac{3}{15}}$; $\sqrt[15]{4^5}$ and $\sqrt[15]{2^3}$, etc.

3. $18^{\frac{1}{2}}$ and $50^{\frac{1}{3}}$; $18^{\frac{3}{6}}$ and $50^{\frac{2}{6}}$; $\sqrt[6]{18^3}$ and $\sqrt[6]{50^2}$, etc.

4. $2^{\frac{1}{m}}$ and $2^{\frac{1}{n}}$; $2^{\frac{n}{mn}}$ and $2^{\frac{m}{mn}}$; $\sqrt[mn]{2^n}$ and $\sqrt[mn]{2^m}$.

5. $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$; $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$; $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$.

6. $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$; $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$, etc.

CIX.

1. $\sqrt{24}=\sqrt{(4\times 6)}=2\sqrt{6}$.

2. $\sqrt{50}=\sqrt{(25\times 2)}=5\sqrt{2}$.

3. $\sqrt{(4a^3)}=2a\sqrt{a}$.

4. $\sqrt{(125a^4d^3)}=5a^2d\sqrt{(5d)}$.

5. $\sqrt{(16\times 2yz^3)}=4z\sqrt{(2yz)}$.

6. $\sqrt{(100\times 10a)}=10\sqrt{(10a)}$.

7. $\sqrt{(144\times 5c^2)}=12c\sqrt{5}$.

8. $7.\sqrt{(36\times 11x)}=42\sqrt{(11x)}$.

9. $18.\sqrt{\left(\frac{1}{9}\times\frac{5}{3}x^3\right)}=6x\sqrt{\frac{5x}{3}}$.

10. $a\sqrt{\left(a^2\times\frac{a}{b}\right)}=a^2\sqrt{\frac{a}{b}}$.

11. $\sqrt{\{a(a^2+2ax+x^2)\}}=(a+x)\sqrt{a}$.

12. $\sqrt{\{x(x^2-2xy+y^2)\}}=(x-y)\sqrt{x}$.

13. $\sqrt{\{2(25a^2-50ab+25b^2)\}}=5(a-b)\sqrt{2}$.

14. $\sqrt{\{7y(9c^4-6c^2y+y^2)\}}=(3c^2-y)\sqrt{(7y)}$.

15. $\sqrt[3]{(27a^6\times 2b^2)}=3a^2\sqrt[3]{(2b^2)}$.

16. $\sqrt[3]{(20xy\times 8x^3y^6)}=2xy^2\sqrt[3]{(20xy)}$.

17. $\sqrt[3]{(27m^9n^9\times 4n)}=3m^3n^3\sqrt[3]{(4n)}$.

18. $\sqrt[3]{(343a^{15}b^{15}\times 4b)}=7a^5b^5\sqrt[3]{(4b)}$.

19. $\sqrt[3]{\{x(x^3+3x^2y+3xy^2+y^3)\}}=(x+y)\sqrt[3]{x}$.

20. $\sqrt[3]{\{a(a^3-3a^2b+3ab^2-b^3)\}}=(a-b)\sqrt[3]{a}$.

CX.

1. $4\sqrt{3}=\sqrt{16}.\sqrt{3}=\sqrt{48}$.

2. $3\sqrt{7}=\sqrt{9}.\sqrt{7}=\sqrt{63}$.

3. $5\sqrt[3]{9}=\sqrt[3]{125}.\sqrt[3]{9}=\sqrt[3]{1125}$.

4. $2\sqrt[4]{6}=\sqrt[4]{16}.\sqrt[4]{6}=\sqrt[4]{96}$.

5. $3\sqrt[3]{\frac{3}{7}}=\sqrt[3]{27}.\sqrt[3]{\frac{3}{7}}=\sqrt[3]{\frac{81}{7}}$.

6. $3\sqrt{a}=\sqrt{9}.\sqrt{a}=\sqrt{(9a)}$.

7. $4a\sqrt{(3x)} = \sqrt{(16a^2)}\sqrt{(3x)} = \sqrt{(48a^2x)}$.

8. $2ax\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(4a^2x^2)}\,.\,\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(3a^3x)}$.

9. $\sqrt{(m+n)^2}.\sqrt{\left(\frac{m-n}{m+n}\right)} = \sqrt{(m^2-n^2)}$.

10. $\sqrt{(a+b)^2}.\sqrt{\left(\frac{1}{a^2-b^2}\right)} = \sqrt{\frac{(a+b)^2}{a^2-b^2}} = \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$.

11. $\sqrt{.\left(\frac{x-y}{x+y}\right)^2}\cdot\sqrt{\left(\frac{x^2+xy}{x^2-2xy+y^2}\right)} = \sqrt{.\frac{x^2+xy}{(x+y)^2}} = \left(\frac{x}{x+y}\right)^{\frac{1}{2}}$.

CXI.

1. $\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$; $\sqrt[3]{4} = \sqrt[6]{4^2} = \sqrt[6]{16}$, etc.

2. $\sqrt{10} = \sqrt[6]{10^3} = \sqrt[6]{1000}$; $\sqrt[3]{15} = \sqrt[6]{15^2} = \sqrt[6]{225}$, etc.

3. $2\sqrt{3} = \sqrt{(4\times3)} = \sqrt{12}$; $3\sqrt{2} = \sqrt{(9\times2)} = \sqrt{18}$, etc.

4. $\sqrt{\frac{3}{5}} = \sqrt[6]{\frac{27}{125}}$; $\sqrt[3]{\frac{14}{15}} = \sqrt[6]{\frac{196}{225}}$, etc.

5. $3\sqrt{7} = \sqrt{(9\times7)} = \sqrt{63}$; $4\sqrt{3} = \sqrt{(16\times3)} = \sqrt{48}$, etc.

6. $2\sqrt{87} = \sqrt{(4\times87)} = \sqrt{348}$; $3\sqrt{33} = \sqrt{(9\times33)} = \sqrt{297}$, etc.

7. $2\sqrt[3]{22} = \sqrt[3]{176} = \sqrt[6]{30976}$; $3\sqrt[3]{7} = \sqrt[3]{189} = \sqrt[6]{35721}$;
$4\sqrt{2} = \sqrt{32} = \sqrt[6]{32768}$, etc.

8. $3\sqrt{19} = \sqrt{171} = \sqrt[6]{5000211}$; $5\sqrt[3]{18} = \sqrt[3]{2250} = \sqrt[6]{5062500}$;
$3\sqrt[3]{82} = \sqrt[3]{2214} = \sqrt[6]{4901796}$, etc.

9. $2\sqrt[3]{14} = \sqrt[3]{112}$; $5\sqrt[3]{2} = \sqrt[3]{250}$; $3\sqrt[3]{3} = \sqrt[3]{81}$, etc.

10. $\frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{4}}$; $\frac{1}{3}\sqrt{3} = \sqrt{\frac{1}{9}}$; $\frac{1}{4}\sqrt{4} = \sqrt{\frac{1}{16}}$, etc.

CXII.

1. $3\sqrt{3}+8\sqrt{3}+18\sqrt{3}=29\sqrt{3}$.

2. $30\sqrt{10}+20\sqrt{2}+144\sqrt{2}=30\sqrt{10}+164\sqrt{2}$.

3. $a^2\sqrt{x}+b^2\sqrt{x}+c^2\sqrt{x}=(a^2+b^2+c^2)\sqrt{x}$.

4. $4\sqrt[3]{2}+7\sqrt[3]{2}+2\sqrt[3]{2}=13\sqrt[3]{2}$. 5. $21\sqrt[3]{2}+6\sqrt[3]{2}+6\sqrt[3]{2}=33\sqrt[3]{2}$.

6. $4\sqrt{6}-3\sqrt{6}=\sqrt{6}$. 7. $9\sqrt{3}-4\sqrt{3}=5\sqrt{3}$.

8. $72\sqrt{2}-24\sqrt{2}=48\sqrt{2}$. 9. $10\sqrt[3]{2}-6\sqrt[3]{2}=4\sqrt[3]{2}$.

10. $21\sqrt[3]{3}-21\sqrt[3]{3}=0$. 11. $\sqrt{6}\times\sqrt{8}=\sqrt{48}=4\sqrt{3}$.

12. $\sqrt{14}\times\sqrt{20}=\sqrt{280}=2\sqrt{70}$. 13. $\sqrt{50}\times\sqrt{200}=\sqrt{10000}=100$.

14. $\sqrt[3]{(3a^2b)}\times\sqrt[3]{(9ab^2)}=\sqrt[3]{(27a^3b^3)}=3ab$.

15. $\sqrt[3]{(12ab)}\times\sqrt[3]{(8a^2b^3)}=\sqrt[3]{(96a^3b^4)}=2ab\sqrt[3]{(12b)}$.

16. $\sqrt{12}\div\sqrt{3}=\sqrt{4}=2$.

17. $\sqrt{18}\div\sqrt{50}=\sqrt{\frac{9}{25}}=\frac{3}{5}$.

18. $\sqrt[3]{(a^2b)}\div\sqrt[3]{(ab^2)}=\sqrt[3]{\left(\frac{a}{b}\right)}$.

19. $\sqrt[4]{(a^3b)}\div\sqrt[4]{(ab^3)}=\sqrt[4]{\left(\frac{a^2}{b^2}\right)}=\sqrt{\left(\frac{a}{b}\right)}$.

20. $\sqrt{(x^2+x^3y)}\div\sqrt{(x+2x^2y+x^3y^2)}=\sqrt{\left(\frac{x+x^2y}{1+2xy+x^2y^2}\right)}=\sqrt{\frac{x}{1+xy}}$.

CXIII.

1. $\sqrt{x}\times\sqrt{y}=\sqrt{(xy)}$. 2. $\sqrt{(x-y)}\times\sqrt{y}=\sqrt{(xy-y^2)}$.

3. $\sqrt{(x+y)}\times\sqrt{(x+y)}=x+y$. 4. $\sqrt{(x-y)}\times\sqrt{(x+y)}=\sqrt{(x^2-y^2)}$.

5. $6\sqrt{x}\times 3\sqrt{x}=18x.$

6. $7\sqrt{(x+1)}\times 8\sqrt{(x+1)}=56(x+1).$

7. $10\sqrt{x}\times 9\sqrt{(x-1)}=90\sqrt{(x^2-x)}.$

8. $\sqrt{(3x)}\times\sqrt{(4x)}=\sqrt{(12x^2)}=2x\sqrt{3}.$

9. $\sqrt{x}\times -\sqrt{x}=-x.$

10. $\sqrt{(x-1)}\times -\sqrt{(x-1)}=-(x-1)=1-x.$

11. $3\sqrt{x}\times -4\sqrt{x}=-12x.$

12. $-2\sqrt{a}\times -3\sqrt{a}=6a.$

13. $\sqrt{(x-7)}\times -\sqrt{x}=-\sqrt{(x^2-7x)}.$

14. $-2\sqrt{(x+7)}\times -3\sqrt{x}=6\sqrt{(x^2+7x)}.$

15. $-4\sqrt{(a^2-1)}\times -2\sqrt{(a^2-1)}=8(a^2-1).$

16. $2\sqrt{(a^2-2a+3)}\times -3\sqrt{(a^2-2a+3)}=-6(a^2-2a+3)=$
$-6a^2+12a-18.$

CXIV.

1.
$$\begin{array}{l} \sqrt{x}+7 \\ \sqrt{x}+2 \\ \hline x+7\sqrt{x} \\ \quad +2\sqrt{x}+14 \\ \hline x+9\sqrt{x}+14 \end{array}$$

2.
$$\begin{array}{l} \sqrt{x}-5 \\ \sqrt{x}+3 \\ \hline x-5\sqrt{x} \\ \quad +3\sqrt{x}-15 \\ \hline x-2\sqrt{x}-15 \end{array}$$

3.
$$\begin{array}{l} \sqrt{(a+9)}+3 \\ \sqrt{(a+9)}-3 \\ \hline a+9+3\sqrt{(a+9)} \\ \qquad -3\sqrt{(a+9)}-9 \\ \hline a \end{array}$$

4.
$$\begin{array}{l} \sqrt{(a-4)}-7 \\ \sqrt{(a-4)}+7 \\ \hline a-4-7\sqrt{(a-4)} \\ \qquad +7\sqrt{(a-4)}-49 \\ \hline a-53 \end{array}$$

5.
$$\begin{array}{rrr}
3\sqrt{x} - & 7 & \\
\sqrt{x} + & 4 & \\
\hline
3x - & 7\sqrt{x} & \\
 & +12\sqrt{x} & -28 \\
\hline
3x + & 5\sqrt{x} & -28
\end{array}$$

6.
$$\begin{array}{lrr}
2\sqrt{(x-5)}+4 & & \\
3\sqrt{(x-5)}-6 & & \\
\hline
6(x-5) & +12\sqrt{(x-5)} & \\
 & -12\sqrt{(x-5)} & -24 \\
\hline
6x-54 & &
\end{array}$$

7.
$$\begin{array}{lll}
\sqrt{(6+x)}+\sqrt{x} & & \\
\sqrt{(6+x)}-\sqrt{x} & & \\
\hline
6+x & +\sqrt{(6x+x^2)} & \\
 & -\sqrt{(6x+x^2)} & -x \\
\hline
6 & &
\end{array}$$

8.
$$\begin{array}{llll}
\sqrt{(3x+1)}+\sqrt{(2x-1)} & & & \\
\sqrt{3x} \quad -\sqrt{(2x-1)} & & & \\
\hline
\sqrt{(9x^2+3x)} & +\sqrt{(6x^2-3x)} & & \\
 & +\sqrt{(6x^2-x-1)} & -(2x-1) & \\
\hline
\sqrt{(9x^2+3x)} & +\sqrt{(6x^2-3x)} & +\sqrt{(6x^2-x-1)} & -2x+1
\end{array}$$

9.
$$\begin{array}{llll}
\sqrt{a}+\sqrt{(a-x)} & & & \\
\sqrt{x}-\sqrt{(a-x)} & & & \\
\hline
\sqrt{(ax)} & +\sqrt{(ax-x^2)} & & \\
 & -\sqrt{(a^2-ax)} & -(a-x) & \\
\hline
\sqrt{(ax)} & +\sqrt{(ax-x^2)} & -\sqrt{(a^2-ax)} & -a+x
\end{array}$$

10.
$$\begin{array}{l}
\sqrt{(3+x)}+\sqrt{x} \\
\sqrt{(3+x)} \\
\hline
3+x+\sqrt{(3x+x^2)}
\end{array}$$

11.
$$\begin{array}{l}
\sqrt{x}+\sqrt{y}+\sqrt{z}\\
\sqrt{x}-\sqrt{y}+\sqrt{z}\\
\hline
x+\sqrt{(xy)}+\sqrt{(xz)}\\
\quad-\sqrt{(xy)}-y-\sqrt{(yz)}\\
\qquad\qquad+\sqrt{(xz)}+\sqrt{(yz)}+z\\
\hline
x-y+2\sqrt{(xz)}+z
\end{array}$$

12.
$$\begin{array}{l}
\sqrt{a}+\sqrt{(a-x)}+\sqrt{x}\\
\sqrt{a}-\sqrt{(a-x)}+\sqrt{x}\\
\hline
a+\sqrt{(a^2-ax)}+\sqrt{(ax)}\\
\quad-\sqrt{(a^2-ax)}-(a-x)-\sqrt{(ax-x^2)}\\
\qquad\qquad\qquad+\sqrt{(ax)}+\sqrt{(ax-x^2)}+x\\
\hline
2x+2\sqrt{(ax)}
\end{array}$$

13.
$$\begin{array}{l}
21+\sqrt{(x^2-9)}\\
21+\sqrt{(x^2-9)}\\
\hline
441+21\sqrt{(x^2-9)}\\
\quad+21\sqrt{(x^2-9)}+x^2-9\\
\hline
432+42\sqrt{(x^2-9)}+x^2
\end{array}$$

14.
$$\begin{array}{l}
\sqrt{(x+3)}+\sqrt{(x+8)}\\
\sqrt{(x+3)}+\sqrt{(x+8)}\\
\hline
x+3+\sqrt{(x^2+11x+24)}\\
\qquad+\sqrt{(x^2+11x+24)}+x+8\\
\hline
2x+11+2\sqrt{(x^2+11x+24)}
\end{array}$$

15.
$$\begin{array}{l}
\sqrt{x}+\sqrt{(x-4)}\\
\sqrt{x}+\sqrt{(x-4)}\\
\hline
x+\sqrt{(x^2-4x)}\\
\quad+\sqrt{(x^2-4x)}+x-4\\
\hline
2x-4+2\sqrt{(x^2-4x)}
\end{array}$$

16.
$$\begin{array}{l}
\sqrt{(x-6)}+\sqrt{x}\\
\sqrt{(x-6)}+\sqrt{x}\\
\hline
x-6+\sqrt{(x^2-6x)}\\
\qquad+\sqrt{(x^2-6x)}+x\\
\hline
2x-6+2\sqrt{(x^2-6x)}
\end{array}$$

17. $$\begin{array}{l} 2\sqrt{x}-3 \\ 2\sqrt{x}-3 \\ \hline 4x-6\sqrt{x} \\ \quad -6\sqrt{x}+9 \\ \hline 4x-12\sqrt{x}+9 \end{array}$$

18. $$\begin{array}{l} \sqrt{(x+y)}-\sqrt{(x-y)} \\ \sqrt{(x+y)}-\sqrt{(x-y)} \\ \hline x+y-\sqrt{(x^2-y^2)} \\ \qquad -\sqrt{(x^2-y^2)}+x-y \\ \hline 2x-2\sqrt{(x^2-y^2)} \end{array}$$

19. $$\begin{array}{l} \sqrt{x}\,.\,\sqrt{(x+1)}-\sqrt{(x-1)} \\ \sqrt{x}\,.\,\sqrt{(x+1)}-.\sqrt{(x-1)} \\ \hline x(x+1)-\sqrt{x}\,.\,\sqrt{(x^2-1)} \\ \qquad -\sqrt{x}\,.\,\sqrt{(x^2-1)}+(x-1) \\ \hline x^2+2x-1-2\sqrt{(x^3-x)} \end{array}$$

20. $$\begin{array}{l} \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \hline x+1+\sqrt{x}.\sqrt{(x^2-1)} \\ \qquad +\sqrt{x}.\sqrt{(x^2-1)}+x(x-1) \\ \hline x^2+1+2\sqrt{(x^3-x)} \end{array}$$

CXV.

1. $c-d=(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$.

2. $c^2-d=(c+\sqrt{d})(c-\sqrt{d})$.

3. $c-d^2=(\sqrt{c}+d)(\sqrt{c}-d)$.

4. $1-y=(1+\sqrt{y})(1-\sqrt{y})$.

5. $1-3x^2=(1+\sqrt{3}.x)(1-\sqrt{3}.x)$.

6. $5m^2-1=(\sqrt{5}.m+1)(\sqrt{5}.m-1)$.

7. $4a^2-3x=\{2a+\sqrt{(3x)}\}\{2a-\sqrt{(3x)}\}$.

8. $9-8n=\{3+2\sqrt{(2n)}\}\{3-2\sqrt{(2n)}\}$.

9. $11n^2-16=(\sqrt{11}.n+4)(\sqrt{11}.n-4)$.

10. $p^2-4r=(p+2\sqrt{r})(p-2\sqrt{r})$.

11. $p-3q^2=(\sqrt{p}+\sqrt{3}.q)(\sqrt{p}-\sqrt{3}.q)$.

12. $a^{2m}-b^n=(a^m+b^{\frac{n}{2}})(a^m-b^{\frac{n}{2}})$.

13. $\frac{1}{a-\sqrt{b}}\times\frac{a+\sqrt{b}}{a+\sqrt{b}}=\frac{a+\sqrt{b}}{a^2-b}$.

14. $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{a+\sqrt{(ab)}}{a-b}$.

15. $\frac{4+3\sqrt{2}}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{12+17\sqrt{2}+12}{9-8}=24+17\sqrt{2}$.

16. $\frac{2}{2-\sqrt{2}}\times\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{4+2\sqrt{2}}{4-2}=\frac{4+2\sqrt{2}}{2}=2+\sqrt{2}$.

17. $\frac{\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{2\sqrt{3}+3}{4-3}=3+2\sqrt{3}$.

18. $\frac{2-\sqrt{2}}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}=\frac{4-4\sqrt{2}+2}{4-2}=\frac{6-4\sqrt{2}}{2}=3-2\sqrt{2}$.

19. $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\times\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}}=\frac{a+2\sqrt{(ax)}+x}{a-x}$.

20. $\frac{1+\sqrt{x}}{1-\sqrt{x}}\times\frac{1+\sqrt{x}}{1+\sqrt{x}}=\frac{1+2\sqrt{x}+x}{1-x}$.

21. $\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}\times\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}$

$$=\frac{a+x+2\sqrt{(a^2-x^2)}+a-x}{a+x-(a-x)}=\frac{2a+2\sqrt{(a^2-x^2)}}{2x}=\frac{a+\sqrt{(a^2-x^2)}}{x}.$$

22. $\frac{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}{\sqrt{(m^2+1)}+\sqrt{(m^2-1)}}\times\frac{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}$

$$=\frac{m^2+1-2\sqrt{(m^4-1)}+m^2-1}{m^2+1-(m^2-1)}=\frac{2m^2-2\sqrt{(m^4-1)}}{2}$$

$$=m^2-\sqrt{(m^4-1)}.$$

23. $\frac{a+\sqrt{(a^2-1)}}{a-\sqrt{(a^2-1)}}\times\frac{a+\sqrt{(a^2-1)}}{a+\sqrt{(a^2-1)}}=\frac{a^2+2a\sqrt{(a^2-1)}+a^2-1}{a^2-(a^2-1)}$

$$=2a^2-1+2a\sqrt{(a^2-1)}.$$

24. $$\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}\times\frac{a+\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}=\frac{a^2+2a\sqrt{(a^2-x^2)}+a^2-x^2}{a^2-(a^2-x^2)}$$

$$=\frac{2a^2-x^2+2a\sqrt{(a^2-x^2)}}{x^2}.$$

CXVI.

1. $$\begin{array}{r}4+\sqrt{(-3)}\\ 4-\sqrt{(-3)}\\ \hline 16+4\sqrt{(-3)}\qquad\qquad\\ -4\sqrt{(-3)}-(-3)\\ \hline 19\qquad\qquad\end{array}$$

2. $$\begin{array}{r}\sqrt{3}-2\sqrt{(-2)}\\ \sqrt{3}+2\sqrt{(-2)}\\ \hline 3-2\sqrt{(-6)}\qquad\qquad\\ +2\sqrt{(-6)}-4\times(-2)\\ \hline 11\qquad\qquad\end{array}$$

3. $$\begin{array}{r}4\sqrt{(-2)}-2\sqrt{2}\qquad\qquad\qquad\\ \tfrac{1}{2}\sqrt{(-2)}-3\sqrt{2}\qquad\qquad\qquad\\ \hline 2\times(-2)-\sqrt{(-4)}\qquad\qquad\\ -12\sqrt{(-4)}+6\times 2\\ \hline 8-13\sqrt{(-4)}=8-26\sqrt{(-1)}\end{array}$$

4. $$\begin{array}{r}\sqrt{(-2)}+\sqrt{(-3)}+\sqrt{(-4)}\qquad\\ \sqrt{(-2)}-\sqrt{(-3)}-\sqrt{(-4)}\qquad\\ \hline -2+\sqrt{6}+2\sqrt{2}\qquad\qquad\qquad\\ -\sqrt{6}-(-3)-2\sqrt{3}\qquad\\ -2\sqrt{2}-2\sqrt{3}-(-4)\\ \hline -2+3-4\sqrt{3}+4=5-4\sqrt{3}\qquad\end{array}$$

5. $$\begin{array}{r}3\sqrt{(-a)}+\sqrt{(-b)}\\ 4\sqrt{(-a)}-2\sqrt{(-b)}\\ \hline -12a+4\sqrt{(ab)}\qquad\\ -6\sqrt{(ab)}+2b\\ \hline -12a-2\sqrt{(ab)}+2b\end{array}$$

6. $$\begin{array}{r}a+\sqrt{(-a)}\qquad\\ a-\sqrt{(-a)}\qquad\\ \hline a^2+a\sqrt{(-a)}\qquad\\ -a\sqrt{(-a)}+a\\ \hline a^2+a\qquad\qquad\qquad\end{array}$$

7. $$\begin{array}{l} a\sqrt{(-a)}+b\sqrt{(-b)} \\ a\sqrt{(-a)}-b\sqrt{(-b)} \\ \hline -a^3+ab\sqrt{(ab)} \\ \qquad\;\; -ab\sqrt{(ab)}+b^3 \\ \hline -a^3 \qquad\qquad\qquad +b^3 \end{array}$$

8. $$\begin{array}{l} a+\beta\sqrt{(-1)} \\ a-\beta\sqrt{(-1)} \\ \hline a^2+a\beta\sqrt{(-1)} \\ \quad\; -a\beta\sqrt{(-1)}+\beta^2 \\ \hline a^2 \qquad\qquad\qquad +\beta^2 \end{array}$$

9. $$\begin{array}{l} 1-\sqrt{(1-e^2)} \\ 1+\sqrt{(1-e^2)} \\ \hline 1-\sqrt{(1-e^2)} \\ \quad +\sqrt{(1-e^2)}-(1-e^2) \\ \hline \qquad\qquad e^2 \end{array}$$

10. $$\begin{array}{l} e^{p\sqrt{(-1)}}+e^{-p\sqrt{(-1)}} \\ e^{p\sqrt{(-1)}}-e^{-p\sqrt{(-1)}} \\ \hline e^{2p\sqrt{(-1)}}+1 \\ \qquad\qquad -1-e^{-2p\sqrt{(-1)}} \\ \hline e^{2p\sqrt{(-1)}} \qquad -e^{-2p\sqrt{(-1)}} \end{array}$$

CXVII.

1. $$\frac{3x+3\sqrt{(xy)}-3\sqrt{(xy)}+3y}{9\sqrt{(xy)}}=\frac{x+y}{3\sqrt{(xy)}}.$$

2. $$\{1+2\sqrt{(-1)}+(-1)\}+\{1-2\sqrt{(-1)}+(-1)\}$$
$$=2\sqrt{(-1)}-2\sqrt{(-1)}=0.$$

3. $$\frac{2\sqrt{(xy)}+2y+2x-2\sqrt{(xy)}}{4\sqrt{(xy)}}=\frac{x+y}{2\sqrt{(xy)}}.$$

4. $$\{1+2\sqrt{(-1)}-1\}-\{1-2\sqrt{(-1)}-1\}=4\sqrt{(-1)}=\sqrt{(-16)}.$$

5. $$\begin{array}{r} x^2+\sqrt{2}.ax+a^2)\,x^4+a^4\,(x^2-\sqrt{2}.ax+a^2 \qquad\qquad\qquad\;\; \\ x^4+\sqrt{2}.ax^3+a^2x^2 \qquad\qquad\qquad\qquad\qquad\qquad \\ \hline -\sqrt{2}.ax^3-a^2x^2+a^4 \qquad\qquad\qquad\quad\;\; \\ -\sqrt{2}.ax^3-2a^2x^2-\sqrt{2}.a^3x \qquad\qquad\quad \\ \hline a^2x^2+\sqrt{2}.a^3x+a^4 \\ a^2x^2+\sqrt{2}.a^3x+a^4 \\ \hline \end{array}$$

6. $$\begin{array}{r l}
m^2-\sqrt{2}.mn+n^2) & m^4+n^4\,(m^2+\sqrt{2}.mn+n^2 \\
 & \underline{m^4-\sqrt{2}.m^3n+m^2n^2} \\
 & \qquad\quad \sqrt{2}.m^3n-m^2n^2+n^4 \\
 & \qquad\quad \underline{\sqrt{2}.m^3n-2m^2n^2+\sqrt{2}.mn^3} \\
 & \qquad\qquad\qquad\quad m^2n^2-\sqrt{2}.mn^3+n^4 \\
 & \qquad\qquad\qquad\quad \underline{m^2n^2-\sqrt{2}.mn^3+n^4}
\end{array}$$

7. $\sqrt{x}.\sqrt{.(x^2+2xy+y^2)}+\sqrt{x}.\sqrt{(x^2-2xy+y^2)}$
$=\sqrt{x}.(x+y)+\sqrt{x}.(x-y)=2x\sqrt{x}.$

8. $$\frac{a\sqrt{a}+a\sqrt{b}-b\sqrt{a}-b\sqrt{b}-(a\sqrt{a}+b\sqrt{a}-a\sqrt{b}-b\sqrt{b})}{a-b}$$
$$=\frac{2a\sqrt{b}-2b\sqrt{a}}{a-b}.$$

9. $a^2.\frac{c}{b}-2a.\sqrt{\frac{c}{b}}\cdot\sqrt{(cd)}+cd=\frac{a^2c}{b}-2ac\sqrt{\frac{d}{b}}+cd.$

10. $a^{2\sqrt{2}}-2a^{\sqrt{2}}.\frac{1}{a^{\sqrt{2}}}+\frac{1}{a^{2\sqrt{2}}}=a^{2\sqrt{2}}-2+\frac{1}{a^{2\sqrt{2}}}.$

11. $$\frac{x^2+a^2+2\sqrt{(x^4-a^4)}+x^2-a^2+x^2+a^2-2\sqrt{(x^4-a^4)}+x^2-a^2}{(x^2+a^2)-(x^2-a^2)}$$
$$=\frac{4x^2}{2a^2}=\frac{2x^2}{a^2}.$$

12. $\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}}\right\}\div\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x^2)}}\right\}=\frac{\sqrt{(1-x^2)}}{\sqrt{(1+x)}}=\sqrt{(1-x)}.$

13. $$\frac{x-1}{x+1}\left\{\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}+\frac{(1+\sqrt{x})(1-\sqrt{x})}{\sqrt{x}(1+\sqrt{x})}\right\}$$
$$=\frac{x-1}{x+1}\left\{\sqrt{x}+1+\frac{1-\sqrt{x}}{\sqrt{x}}\right\}=\frac{x-1}{x+1}\left(\frac{x+\sqrt{x}+1-\sqrt{x}}{\sqrt{x}}\right)=\frac{x-1}{\sqrt{x}}.$$

14. $\frac{x}{4}+3-2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{x}{2}-2\sqrt{\left(\frac{x^2}{16}-9\right)}.$

15. $x+a-2\sqrt{(x^2-a^2)}+x-a=2x-2\sqrt{(x^2-a^2)}$.

16. $\sqrt[m]{(a^{2m-n+n}.b^{5m+1+m-1}.c^{3p+m-3p})}=\sqrt[m]{(a^{2m}.b^{6m}.c^m)}=a^2b^6c$.

17. $\{-1-a\sqrt{(-1)}\}^2=1+2a\sqrt{(-1)}-a^2$

$$\{1+2a\sqrt{(-1)}-a^2\}^2=1-4a^2+a^4+4a\sqrt{(-1)}-2a^2-4a^3\sqrt{(-1)}$$

$$\begin{array}{l}1-6a^2+a^4+4a\sqrt{(-1)}-4a^3\sqrt{(-1)}\\ -1-a\sqrt{(-1)}\\ \hline -1+6a^2-a^4-4a\sqrt{(-1)}+4a^3\sqrt{(-1)}\\ \quad -a\sqrt{(-1)}+6a^3\sqrt{(-1)}-a^5\sqrt{(-1)}+4a^2-4a^4\\ \hline -1+10a^2-5a^4+(10a^3-a^5-5a)\sqrt{(-1)}\end{array}$$

18. $3\sqrt[3]{3}-(-8)+4\sqrt[3]{3}=8+7\sqrt[3]{3}$.

19. $\dfrac{6c^2}{x-1}\cdot\sqrt{\left(\dfrac{x(4x^2-8x+4)}{3c^3}\right)}=\dfrac{6c^2}{x-1}\cdot\dfrac{2x-2}{c}\cdot\sqrt{\dfrac{x}{3c}}=\dfrac{12c\sqrt{x}}{\sqrt{(3c)}}=4\sqrt{(3cx)}$.

20. $\dfrac{x}{x-7}\{\sqrt[3]{[3p^2\{x^3-21x^2+147x-343\}]}\}=\dfrac{x}{x-7}\cdot\sqrt[3]{(3p^2)}.(x-7)$

$=x\sqrt[3]{(3p^2)}$.

21. $2(n-1)\sqrt[3]{\left[\dfrac{1}{2n}\cdot\left(-\dfrac{1}{n^3-3n^2+3n-1}\right)\right]}=2(n-1)\sqrt[3]{\left(\dfrac{1}{2n}\right)\cdot\left(-\dfrac{1}{n-1}\right)}$

$=-2\sqrt[3]{\dfrac{1}{2n}}=-\sqrt[3]{\dfrac{8}{2n}}=-\sqrt[3]{\dfrac{4n^2}{n^3}}=\dfrac{1}{n}\sqrt[3]{(-4n^2)}$.

22. $2(n-1).3\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{2(n-1)\times 5}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=6n\sqrt{7}-6\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{10n}{3}\sqrt{7}-\dfrac{10}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=9n\sqrt{7}-10\sqrt{7}$.

23. $\sqrt{(17^2-33)}\quad\sqrt[3]{(65^2-129)}=\sqrt{256}-\sqrt[3]{4096}=16-16=0$.

CXVIII.

Adopt the notation of Art. 316; then

1. $\left.\begin{array}{r} x+y=10 \\ xy=21 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=100 \\ \quad 4xy \quad = \ 84 \end{array}\right\}; x^2-2xy+y^2=16; x-y=4$, etc.

2. $\left.\begin{array}{r} x+y=16 \\ xy=55 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=256 \\ \quad 4xy \quad =220 \end{array}\right\} x^2-2xy+y^2=36; x-y=6$, etc.

3. $\left.\begin{array}{r} x+y=\ 9 \\ xy=14 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=81 \\ \quad 4xy \quad =56 \end{array}\right\}; x^2-2xy+y^2=25; x-y=5$, etc.

4. $\left.\begin{array}{r} x+y=\ 94 \\ xy=2205 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=8836 \\ \quad 4xy \quad =8820 \end{array}\right\}; x^2-2xy+y^2=16;$
   $x-y=4$, etc.

5. $\left.\begin{array}{r} x+y=13 \\ xy=30 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=169 \\ \quad 4xy \quad =120 \end{array}\right\}; x^2-2xy+y^2=49; x-y=7$, etc.

6. $\left.\begin{array}{r} x+y=\ 38 \\ xy=360 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=1444 \\ \quad 4xy \quad =1440 \end{array}\right\}; x^2-2xy+y^2=4; x-y=2$, etc.

7. $\left.\begin{array}{r} x+y=14 \\ xy=24 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=196 \\ \quad 4xy \quad = \ 96 \end{array}\right\}; x^2-2xy+y^2=100;$
   $x-y=10$, etc.

8. $\left.\begin{array}{r} x+y=103 \\ xy=396 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=10609 \\ \quad 4xy \quad = \ 1584 \end{array}\right\}; x^2-2xy+y^2=9025;$
   $x-y=95$, etc.

9. $\left.\begin{array}{r} x+y=\ 75 \\ xy=756 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=5625 \\ \quad 4xy \quad =3024 \end{array}\right\}; x^2-2xy+y^2=2601;$
   $x-y=51$, etc.

10. $\left.\begin{array}{r} x+y=\ 87 \\ xy=1512 \end{array}\right\}; \left.\begin{array}{l} x^2+2xy+y^2=7569 \\ \quad 4xy \quad =6048 \end{array}\right\}; x^2-2xy+y^2=1521;$
    $x-y=39$, etc.

11. $\left.\begin{aligned} x+y&=\frac{7}{2} \\ xy&=\frac{10}{4} \end{aligned}\right\}; \quad \left.\begin{aligned} x^2+2xy+y^2&=\frac{49}{4} \\ 4xy\quad&=\frac{40}{4} \end{aligned}\right\}; \; x^2-2xy+y^2=\frac{9}{4};$
$x-y=\frac{3}{2}$, etc.

12. $\left.\begin{aligned} x+y&=\ 57 \\ xy&=540 \end{aligned}\right\} \quad \left.\begin{aligned} x^2+2xy+y^2&=3249 \\ 4xy\quad&=2160 \end{aligned}\right\}; \; x^2-2xy+y^2=1089;$
$x-y=33$, etc.

CXIX.

1. $x=7^2=49$.
2. $x=9^2=81$.
3. $x=5^2=25$.
4. $x=2^3=8$.
5. $x=3^3=27$.
6. $x=4^4=256$.
7. $x+9=36$, etc.
8. $x-7=49$, etc.
9. $x-15=64$, etc.
10. $x-9=144$, etc.
11. $4x-16=8$, etc.
12. $3\sqrt{x}=18$; $x=36$.
13. $\sqrt[3]{(2x+3)}=3$; $2x+3=27$; $2x=24$; $x=12$.
14. $c\sqrt{x}=a-b$; $c^2x=(a-b)^2$; $x=\frac{(a-b)^2}{c^2}$.
15. $\sqrt{(x^2-9)}=9-x$; $x^2-9=81-18x+x^2$; $18x=90$; $x=5$.
16. $x^2-11=x^2-2x+1$; $2x=12$; $x=6$.
17. $4x^2+5x-2=4x^2+4x+1$; $x=3$.
18. $9x^2-12x-51=9x^2-18x+9$; $6x=60$; $x=10$.
19. $x^2-ax+b=x^2+2ax+a^2$; $3ax=b-a^2$, etc.
20. $25x^2-3mx+n=m^2+10mx+25x^2$; $13mx=n-m^2$, etc.

CXX.

1. $\sqrt{(16+x)}=8-\sqrt{x}$; $16+x=64-16\sqrt{x}+x$; $16\sqrt{x}=48$; $\sqrt{x}=3$;
$x=9$.

2. $x-16=64-16\sqrt{x}+x$; $16\sqrt{x}=80$; $\sqrt{x}=5$; $x=25$.

3. $\sqrt{(x+15)}=15-\sqrt{x}$; $x+15=225-30\sqrt{x}+x$; $30\sqrt{x}=210$;
 $\sqrt{x}=7$; $x=49$.

4. $x-21=x-2\sqrt{x}+1$; $2\sqrt{x}=22$; $\sqrt{x}=11$; $x=121$.

5. $x-1=x+4-6\sqrt{(x+4)}+9$; $6\sqrt{(x+4)}=14$; $\sqrt{(x+4)}=\frac{7}{3}$;
 $x+4=\frac{49}{9}$; $x=\frac{13}{9}$.

6. $1+2\sqrt{(3x+1)}+3x+1=4x+4$; $2\sqrt{(3x+1)}=x+2$;
 Squaring both sides, $4(3x+1)=x^2+4x+4$; $x^2=8x$, etc.

7. $1-2\sqrt{(1-3x)}+1-3x=4(1-x)$; $2\sqrt{(1-3x)}=x-2$;
 $4(1-3x)=x^2-2x+4$; $x^2+8x=0$, etc.

8. $a-\sqrt{x}=\sqrt{(x-a)}$; $a^2-2a\sqrt{x}+x=x-a$; $2\sqrt{x}=1+a$;
 $4x=(1+a)^2$, etc.

9. $\sqrt{(x-m)}=\frac{m-2\sqrt{x}}{2}$; $x-m=\frac{m^2-4m\sqrt{x}+4x}{4}$; $-4m=m^2-4m\sqrt{x}$;
 $\sqrt{x}=\frac{m+4}{4}$; $x=\left(\frac{m+4}{4}\right)^2$.

10. $\sqrt{(x-4)}=3-\sqrt{(x-1)}$; $x-4=9-6\sqrt{(x-1)}+x-1$;
 $6\sqrt{(x-1)}=12$; $\sqrt{(x-1)}=2$; $x-1=4$; $x=5$.

CXXI.

1. $\sqrt{(x^2-9x)}+x-9=36$; $\sqrt{(x^2-9x)}=45-x$;
 $x^2-9x=2025-90x+x^2$; $81x=2025$; $x=25$.

2. $x+\sqrt{(x^2-21x)}=35$; $x^2-21x=1225-70x+x^2$; $49x=1225$;
 $x=25$.

3. $x+7+\sqrt{(x^2+7x)}=28$; $\sqrt{(x^2+7x)}=21-x$;
 $x^2+7x=441-42x+x^2$; $49x=441$; $x=9$.

4. $x - 15 + \sqrt{(x^2 - 15x)} = 105$; $\sqrt{(x^2 - 15x)} = 120 - x$;
$x^2 - 15x = 14400 - 240x + x^2$; $225x = 14400$; $x = 64$.

5. $\sqrt{(x^2 - 4x)} + x - 4 = 8$; $\sqrt{(x^2 - 4x)} = 12 - x$; $x^2 - 4x = 144 - 24x + x^2$;
$20x = 144$, etc.

6. $\sqrt{(3ax + x^2)} + 3a + x - 9a = 0$; $\sqrt{(3ax + x^2)} = 6a - x$;
$3ax + x^2 = 36a^2 - 12ax + x^2$; $15ax = 36a^2$, etc.

7. $b^2 - ax = b^2 - ax - ab + bx$; $bx = ab$; $x = a$.

8. $2 + \sqrt{x} - x = \frac{4 + \sqrt{x}}{2}$; $4 + 2\sqrt{x} - 2x = 4 + \sqrt{x}$; $\sqrt{x} = 2x$; $x = 4x^2$;
whence $x = 0$ or $\frac{1}{4}$.

9. $x + 28\sqrt{x} + 192 = x + 36\sqrt{x} + 128$; $8\sqrt{x} = 64$; $x = 64$.

10. $x - 6\sqrt{x} - 16 = x - 10\sqrt{x} + 24$; $4\sqrt{x} = 40$; $x = 100$.

CXXII.

1. $x - 3\sqrt{x} = 4$; $3\sqrt{x} = x - 4$; $9x = x^2 - 8x + 16$; $x^2 - 17x = -16$, etc.

2. $14\sqrt{x} = x + 45$; $196x = x^2 + 90x + 2205$; $x^2 - 106x = -2205$, etc.

3. $9(7 + 2x^3) = 25.(4x - 3)$; $63 + 18x^2 = 100x - 75$;
$18x^2 - 100x = -138$, etc.

4. $6x - 11 = 249 - 2x^2$; $2x^2 + 6x = 260$; $x^2 + 3x = 130$, etc.

5. $6 - x = 4 - 4\sqrt{(2x - 1)} + 2x - 1$; $4\sqrt{(2x - 1)} = 3x - 3$;
$16(2x - 1) = 9x^2 - 18x + 9$; $9x^2 - 50x = -25$, etc.

6. $x + 12 = 2\sqrt{(4 - 3x)}$; $x^2 + 24x + 144 = 4(4 - 3x)$; $x^2 + 36x = -128$.

7. $2x + 7 + 2\sqrt{(6x^2 - 15x - 126)} + 3x - 18 = 7x + 1$;
$\sqrt{(6x^2 - 15x - 126)} = x + 6$; $6x^2 - 15x - 126 = x^2 + 12x + 36$, etc.

8. $4(204-5x)=400-40\sqrt{(3x-68)}+3x-68$;
$40\sqrt{(3x-68)}=23x-484$;
$1600(3x-68)-529x^2-22264x+234256$;
$529x^2-27064x=-343056$;
$x^2-\frac{27064}{529}x+\frac{183115024}{(529)^2}=\frac{183115024-181476624}{(529)^2}=\frac{1638400}{(529)^2}$
$x-\frac{13532}{529}=\pm\frac{1280}{529}$, etc.

9. $(\sqrt{x}-4)(\sqrt{x}+4)=33$; $x-16=33$; $x=49$.

10. $(\sqrt{x}+11).(\sqrt{x}-11)=608$; $x-121=608$; $x=729$.

11. $\sqrt{(x^2+17x+60)}=12$; $x^2+17x+60=144$; $x^2+17x=-84$, etc.

12. $x+3+2\sqrt{(x^2+11x+24)}+x+8=25x$; $2\sqrt{(x^2+11x+24)}=23x-11$;
$4(x^2+11x+24)=529x^2-506x+121$; $525x^2-550x=-25$;
$x^2-\frac{22x}{21}=-\frac{1}{21}$, etc.

13. $\sqrt{(25+x)}=8-\sqrt{(25-x)}$; $25+x=64-16\sqrt{(25-x)}+25-x$;
$8\sqrt{(25-x)}=32-x$; $64(25-x)=1024-64x+x^2$; $x^2=576$;
$x=\pm 24$.

14. $x+4+2\sqrt{(2x^2+7x-4)}+2x-1=36$; $2\sqrt{(2x^2+7x-4)}=33-3x$,
$4(2x^2+7x-4)=1089-198x+9x^2$; $x^2-226x=-1105$, etc.

15. $\sqrt{(13x-1)}=5+\sqrt{(2x-1)}$; $13x-1=25+10\sqrt{(2x-1)}+2x-1$;
$11x-25=10\sqrt{(2x-1)}$; $121x^2-550x+625=100(2x-1)$;
$121x^2-750x=-725$; $x^2-\frac{750}{121}x+\frac{140625}{(121)^2}=\frac{140625-87725}{(121)^2}$, etc.

16. $7x+1-2\sqrt{(21x^2+10x+1)}+3x+1=4$; $\sqrt{(21x^2+10x+1)}=5x-1$;
$21x^2+10x+1=25x^2-10x+1$; $4x^2=20x$; $x=5$ or 0.

17. $\sqrt{(4+x)}=3-\sqrt{x}$; $4+x=9-6\sqrt{x}+x$; $\sqrt{x}=\frac{5}{6}$; $x=\frac{25}{36}$.

18. $x+\sqrt{(x^2+9975)}=525$; $\sqrt{(x^2+9975)}=525-x$;
$x^2+9975x=275625-1050x+x^2$; $11025x=275625$, etc.

19. $\frac{x}{4}+3+2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{2x}{3}$; $2\sqrt{\left(\frac{x^2}{16}-9\right)}=\frac{x}{6}$;
$\frac{x^2}{16}-9=\frac{x^2}{144}$; $\frac{8x^2}{144}=9$; $x^2=9\times 18$; $x=\pm 9\sqrt{2}$.

20. $x^2-1+6\sqrt{(x^2-1)}=16$; $x^2-1+6\sqrt{(x^2-1)}+9=25$;
$\sqrt{(x^2-1)}+3=\pm 5$; $\sqrt{(x^2-1)}=2$ or -8.
Hence $x^2-1=4$ or 64; $x^2=5$ or 65, etc.

21. $x^2-2ax+a^2+2ab+b^2=x^2+a^2+b^2-2ax+2bx-2ab$; $4ab=2bx$;
$x=2a$.

22. $x^2+2ax+a^2+2ab+b^2=b^2+a^2+x^2-2ab-2bx+2ax$; $4ab=-2bx$;
$x=-2a$.

23. $x+4-2\sqrt{(x^2+4x)}+x=x+\frac{3}{2}$; $x+\frac{5}{2}=2\sqrt{(x^2+4x)}$;
$x^2+5x+\frac{25}{4}=4x^2+16x$; $3x^2+11x=\frac{25}{4}$; $x^2+\frac{11x}{3}+\frac{121}{36}=\frac{196}{36}$;
$x+\frac{11}{6}=\pm\frac{14}{6}$; $x=\frac{1}{2}$ or $-\frac{25}{6}$.

24. $\sqrt{x}+1=x+\frac{5}{4}$; $\sqrt{x}=x+\frac{1}{4}$; $x-\sqrt{x}=-\frac{1}{4}$; $x-\sqrt{x}+\frac{1}{4}=0$;
$\sqrt{x}-\frac{1}{2}=0$; $x=\frac{1}{4}$.

25. $\sqrt{(4+x)}=\sqrt{3}+\sqrt{x}$; $4+x=3+2\sqrt{(3x)}+x$; $2\sqrt{(3x)}=1$;
$12x=1$, etc.

26. $\sqrt{(x+4)}=9-\sqrt{(x+5)}$; $x+4=81-18\sqrt{(x+5)}+x+5$;
$18\sqrt{(x+5)}=82$; $9\sqrt{(x+5)}=41$; $81(x+5)=1681$, etc.

27. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-24x+x^2$;
$20x=144$, etc.

28. $x^2-21=\sqrt{(x^2-9)}$; $x^4-42x^2+441=x^2-9$; $x^4-43x^2=-450$;
$x^4-43x^2+\frac{1849}{4}=\frac{49}{4}$; $x^2-\frac{43}{2}=\pm\frac{7}{2}$; $x^2=25$ or 18, etc.

29. $\surd(50+x)=2+\surd(50-x)$; $50+x=4+4\surd(50-x)+50-x$;
$x-2=2\surd(50-x)$; $x^2-4x+4=4(50-x)$; $x^2=196$, etc.

30. $\surd(2x+4)-1=\surd\left(\frac{x}{2}+6\right)$; $2x+4-2\surd(2x+4)+1=\frac{x}{2}+6$;
$\frac{3x}{2}-1=2\surd(2x+4)$; $9x^2-12x+4=16(2x+4)$; $9x^2-44x=60$;
$x^2-\frac{44}{9}x+\frac{484}{81}=\frac{1024}{81}$, etc.

31. $3+x+\surd(3x+x^2)=6$; $\surd(3x+x^2)=3-x$; $3x+x^2=9-6x+x^2$;
$9x=9$; $x=1$.

32. $\surd(x-1)+\surd(x+1)=1$; $\surd(x-1)=1-\surd(x+1)$;
$x-1=1-2\surd(x+1)+x+1$; $2\surd(x+1)=3$; $4(x+1)=9$, etc.

33. $3x+\surd(4x-x^2)=6x-2\surd(4x-x^2)$; $3\surd(4x-x^2)=3x$; $4x-x^2=x^2$;
$2x^2=4x$, etc.

34. $\surd\{a-\surd(ax+x^2)\}=\surd a-\surd x$; $a-\surd(ax+x^2)=a-2\surd(ax)+x$;
$\surd(ax+x^2)=2\surd(ax)-x$; $ax+x^2=4ax-4x\surd(ax)+x^2$;
$4x\surd(ax)=3ax$; whence $x=0$, or $4\surd(ax)=3a$; $16ax=9a^2$;
$16x=9a$, etc.

CXXIII.

1. $x-2=0$, or, $x-5=0$; $\therefore x=2$ or 5.

2. $x-3=0$, or, $x+7=0$; $\therefore x=3$ or -7.

3. $x+9=0$, or, $x+2=0$; $\therefore x=-9$ or -2.

4. $x-5a=0$, or, $x-6b=0$; $\therefore x=5a$ or $6b$.

5. $2x+7=0$, or, $3x-5=0$; $\therefore x=-\frac{7}{2}$ or $\frac{5}{3}$.

6. $19x-227=0$, or, $14x+83=0$; $\therefore x=\frac{227}{19}$ or $-\frac{83}{14}$.

7. $5x-4m=0$, or, $6x-11n=0$; $\therefore x=\frac{4m}{5}$ or $\frac{11n}{6}$.

8. $x^2+5ax+6a^2=0$, or, $x^2-7ax+12a^2=0$;
$(x+2a)(x+3a)=0$, or, $(x-4a)(x-3a)=0$;
$\therefore x+2a=0$, or, $x+3a=0$, or, $x-4a=0$, or, $x-3a=0$, etc.

9. $(x+2)(x-2)(x-a)(x-a)=0$; $\therefore x=\pm 2$ or a.

10. $x.x.(x-5)=0$, $\therefore x=0$ or 5.

11. $acx-2a+b=0$, or, $bcx+3a-b=0$;
$\therefore acx=2a-b$, or, $bcx=b-3a$, etc.

12. $cx-d=0$, or, $cx-e=0$, etc.

CXXIV.

1. Let m be one of the roots of the first equation, and $\frac{1}{m}$ one of the roots of the second equation.
Then $am^2+bm+c=0$, and $c'm^2+b'm+a'=0$.
Multiply the first by a' and the second by c; then
$aa'm^2+a'bm+a'c=0$, and $cc'm^2+b'cm+a'c=0$.
Subtracting $(aa'-cc')m^2+(a'b-b'c)m=0$.
Whence $(aa'-cc')m=-(a'b-b'c)$ . . . . (1).
Again, multiply the first by c' and the second by a; then
$ac'm^2+bc'm+cc'=0$, and $ac'm^2+ab'm+aa'=0$.
Subtracting $(bc'-ab')m=cc'-aa'$;
or $aa'-cc'=-(ab'-bc')m$ . . . . (2).
Multiplying (1) and (2) together, we get
$(aa'-cc')^2=(ab'-bc')(a'b-b'c)$.

2. $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$;

$\therefore \alpha^2+2\alpha\beta+\beta^2=\frac{b^2}{a^2}$, and $2\alpha\beta=\frac{2c}{a}$

$\therefore \alpha^2+\beta^2=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$.

3. $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

$$\therefore ac\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=ac\left(x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1\right)$$
$$=ac\left(x^2-\frac{b^2-2ac}{ac}x+1\right)$$
$$=acx^2+(2ac-b^2)x+ac.$$

4. The roots are $\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$, and that these may be *equal*, we must have $b^2-4ac=0$, or $c=\dfrac{b^2}{4a}$.

Putting this for c in the expression ax^2+bx+c,

$$ax^2+bx+\frac{b^2}{4a}=\frac{4a^2x^2+4abx+b^2}{4a}=\left(\frac{2ax+b}{2\sqrt{a}}\right)^2.$$

5. $\alpha+\beta=1+a$, and $\alpha\beta=\dfrac{1+a+a^2}{2}$.

$\alpha^2+2\alpha\beta+\beta^2=1+2a+a^2$, and $2\alpha\beta=1+a+a^2$

$\therefore \alpha^2+\beta^2=a.$

CXXV.

1. $(x-5)(x-6)=0$, or, $x^2-11x+30=0$.

2. $(x-4)(x+5)=0$, or, $x^2+x-20=0$.

3. $(x+2)(x+7)=0$, or, $x^2+9x+14=0$.

4. $\left(x-\dfrac{1}{2}\right)\left(x-\dfrac{2}{3}\right)=0$, or, $(2x-1)(3x-2)=0$, or, $6x^2-7x+2=0$.

5. $(x-7)\left(x+\dfrac{5}{9}\right)=0$, or, $(x-7)(9x+5)=0$, or, $9x^2-58x-35=0$.

6. $(x-\sqrt{3})(x+\sqrt{3})=0$, or, $x^2-3=0$.

7. $(x-m-n)(x-m+n)=0$, or, $x^2-2mx+m^2-n^2=0$.

8. $\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)=0$, or, $x^2-\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0$.

9. $\left(x+\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$, or, $x^2+\frac{\alpha^2-\beta^2}{\alpha\beta}x-1=0$.

CXXVI.

1. One root is found by trial to be 2 ; then
$$x^3-11x^2+36x-36=(x-2)(x^2-9x+18)=(x-2)(x-3)(x-6).$$

2. One root is found by trial to be 1 ; then
$$x^3-7x^2+14x-8=(x-1)(x^2-6x+8)=(x-1)(x-2)(x-4).$$

3. One root is found by trial to be -1 ; then
$$x^3-5x^2-46x-40=(x+1)(x^2-6x-40)=(x+1)(x-10)(x+4).$$

4. One root is found by trial to be -1 ; then
$$4x^3+6x^2+x-1=(x+1)(4x^2+2x-1)=(x+1)4.\left(x^2+\frac{x}{2}-\frac{1}{4}\right)$$
and since the roots of $x^2+\frac{x}{2}-\frac{1}{4}=0$ are $\frac{\sqrt{5}-1}{4}$ and $\frac{-\sqrt{5}-1}{4}$,
$$x^2+\frac{x}{2}-\frac{1}{4}=\left(x+\frac{1-\sqrt{5}}{4}\right).\left(x+\frac{1+\sqrt{5}}{4}\right),$$ and therefore
$$4x^3+6x^2+x-1=4(x+1)\left(x+\frac{1-\sqrt{5}}{4}\right)\left(x+\frac{1+\sqrt{5}}{4}\right).$$

5. One root is found by trial to be -1 ; then
$$6x^3+11x^2-9x-14=(x+1)(6x^2+5x-14)=(x+1)(x+2)(6x-7).$$

6. If we put $x=-y-z$, the expression becomes zero, which shows that $x+y+z$ is one of its factors ; the other is found by division to be $x^2+y^2+z^2-xy-xz-yz$.

7. If we put $a=b+c$, the expression becomes zero, which shows that $a-b-c$ is one of its factors ; the other is found by division to be $a^2+b^2+c^2+ab+ac-bc$.

8. One root is found by trial to be 1; then
$3x^3-x^2-23x+21=(x-1)(3x^2+2x-21)=(x-1)(3x-7)(x+3).$

9. One root is found by trial to be 1; then
$2x^3-5x^2-17x+20=(x-1)(2x^2-3x-20)=(x-1)(x-4)(2x+5).$

10. One root is found by trial to be -1; then
$15x^3+41x^2+5x-21=(x+1)(15x^2+26x-21)$
$=(x+1)(3x+7)(5x-3).$

CXXVII.

1. $x^4-12x^2=13$; $x^4-12x^2+36=49$; $x^2-6=\pm 7$, etc.

2. $x^6+14x^3=-24$; $x^6+14x^3+49=25$; $x^3+7=\pm 5$, etc.

3. $x^8+22x^4=-21$; $x^8+22x^4+121=100$; $x^4+11=\pm 10$, etc.

4. $x^{2m}+3x^m+\frac{9}{4}=\frac{25}{4}$; $x^m+\frac{3}{2}=\pm\frac{5}{2}$, etc.

5. $x^{4n}-\frac{5}{3}x^{2n}+\frac{25}{36}=\frac{100}{36}$; $x^{2n}-\frac{5}{6}=\pm\frac{10}{6}$, etc.

6. $x-\frac{9}{2}x^{\frac{1}{2}}+\frac{81}{16}=\frac{121}{16}$; $x^{\frac{1}{2}}-\frac{9}{4}=\pm\frac{11}{4}$, etc.

7. $x^{-2}+3x^{-1}+\frac{9}{4}=\frac{97}{36}$; $x^{-1}+\frac{3}{2}=\pm\frac{\sqrt{97}}{6}$; $x^{-1}=\frac{-9\pm\sqrt{97}}{6}$
$\therefore x=\frac{6}{-9\pm\sqrt{97}}.$

8. $x^{-2n}-x^{-n}+\frac{1}{4}=\frac{81}{4}$; $x^{-n}-\frac{1}{2}=\pm\frac{9}{2}$; $x^{-n}=5$ or -4,
$\therefore \frac{1}{x^n}=\frac{1}{5}$ or $-\frac{1}{4}$; $x=\left(\frac{1}{5}\right)^{\frac{1}{n}}$ or $\left(-\frac{1}{4}\right)^{\frac{1}{n}}.$

9. $x^2-2x+5+6(x^2-2x+5)^{\frac{1}{2}}=16$;
$(x^2-2x+5)+6(x^2-2x+5)^{\frac{1}{2}}+9=25$; $(x^2-2x+5)^{\frac{1}{2}}+3=\pm 5$;
$(x^2-2x+5)^{\frac{1}{2}}=2$ or -8; $x^2-2x+5=4$ or 64, etc.

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;

$$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39;$$
$$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=64;$$
$$\sqrt{(2x^2-5x+6)}+5=\pm 8;\ \sqrt{(2x^2-5x+6)}=3 \text{ or } -13;$$
$$2x^2-5x+6=9 \text{ or } 169, \text{ etc.}$$

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;

$$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8;$$
$$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=a^2+2a+1;$$
$$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1);$$
$$3x^2-2ax+4=(4+a)^2 \text{ or } (2-a)^2, \text{ etc.}$$

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;

$$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-2a+1;$$
$$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1);\ x^2-ax+a^2=a^2 \text{ or } (2-a)^2, \text{ etc.}$$

CXXVIII.

1. $\frac{2}{3}, \frac{6}{7}, \frac{7}{9};\ \frac{42}{63}, \frac{54}{63}, \frac{49}{63}$, etc.

2. $\frac{x+3y}{x+2y}, \frac{x+2y}{x+y};\ \frac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \frac{x^2+4xy+4y^2}{(x+2y)(x+y)}$, etc.

3. $\frac{x-5y}{x-4y}, \frac{x-3y}{x-2y};\ \frac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \frac{x^2-7xy+12y^2}{(x-4y)(x-2y)}$, etc.

4. Let x be the number.

Then $\frac{a+x}{b+x}=\frac{c}{d}$; $ad+dx=bc+cx$;

$(c-d)x=ad-bc$; $x=\frac{ad-bc}{c-d}$.

5. Let $x:y$ be the ratio. Then

$$\left.\begin{aligned}x^2+y^2&=181\\ xy&=90\end{aligned}\right\},\ \left.\begin{aligned}x^2+y^2+2xy&=361\\ x^2+y^2-2xy&=1\end{aligned}\right\},\ \left.\begin{aligned}x+y&=19\\ x-y&=\pm 1\end{aligned}\right\},\ \begin{aligned}x&=10 \text{ or } 9\\ y&=9 \text{ or } 10\end{aligned}.$$

CXXIX.

1. $x:y=6:9=2:3$. 2. $x:y=b:a$.

3. $(a-c)x=(b+d)y$; $x:y=b+d:a-c$.

4. $x^2+2xy+y^2=6y^2$; $x+y=\pm\sqrt{6}.y$;

$x=(\pm\sqrt{6}-1)y$; $x:y=\pm\sqrt{6}-1:1$.

5. $x^2-12xy+36y^2=49y^2$; $x-6y=\pm 7y$;

$x=13y$ or $-y$; $x:y=13:1$, or, $-1:1$.

6. $x^2+mxy+\frac{m^2y^2}{4}=\frac{(m^2+4n^2)y^2}{4}$; $x+\frac{my}{2}=\frac{\pm\sqrt{(m^2+4n^2)}y}{2}$;

$x=\frac{-m\pm\sqrt{(m^2+4n^2)}}{2}y$, etc.

7. Let $3x$ and $4x$ be the numbers. Then

$\frac{3x+4x}{9x^2+16x^2}=\frac{7}{50}$; $7\times 25x^2=50\times 7x$; $x=2$.

Hence the numbers are 6 and 8.

8. Let $6x$ and $7x$ be the numbers. Then

$\frac{6x+12}{7x+12}=\frac{12}{13}$; $78x+156=84x+144$; $6x=12$; $x=2$, etc.

9. Let $7x$ and $13x$ be the numbers. Then $20x=100$; $x=5$, etc.

10. Let x and y be the numbers. Then $x^2-y^2=48$, and $\frac{x+y}{x-y}=\frac{12}{1}$.

Hence $x+y=12x-12y$, or, $13y=11x$.

Then $x^2-\frac{121x^2}{169}=48$; $48x^2=169\times 48$; $x=13$, etc.

11. Let x be the value of a gold coin, and y the value of a silver one. Then $5x+4y=3x+12y$; $2x=8y$; $x:y=4:1$.

12. Let x be the value of a gold coin, and y the value of a silver one. Then $8x+9y=6x+19y$; $2x=10y$; $y:x=1:5$.

CXXX.

1. $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$. 2. $\frac{3}{7}\times\frac{14}{9}\times\frac{4}{3}=\frac{8}{9}$.

3. $\frac{x^2-y^2}{x^3+y^3}\times\frac{x^2-xy+y^2}{x+y}=\frac{x-y}{x+y}$.

4. $\frac{a^2-b^2+2bc-c^2}{a^2-b^2-2bc-c^2}\times\frac{a+b+c}{a+b-c}=\frac{(a+b-c)(a-b+c)(a+b+c)}{(a+b+c)(a-b-c)(a+b-c)}$, etc.

5. $\frac{m^3+n^3}{m^3-n^3}\times\frac{m-n}{m+n}=\frac{m^2-mn+n^2}{m^2+mn+n^2}$.

6. $\frac{(x+2)(x+3)}{(y-3)(y-4)}\times\frac{y(y-3)}{x(x+3)}=\frac{(x+2)y}{(y-4)x}$.

CXXXI.

1. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$ and $c=\lambda d$.

Then $\frac{a+b}{a}=\frac{\lambda b+b}{\lambda b}=\frac{\lambda+1}{\lambda}$,

and $\frac{c+d}{c}=\frac{\lambda d+d}{\lambda d}=\frac{\lambda+1}{\lambda}$.

2. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$, $c=\lambda d$.

Then $\frac{a^2-b^2}{b^2}=\frac{\lambda^2b^2-b^2}{b^2}=\lambda^2-1$,

and $\frac{c^2-d^2}{d^2}=\frac{\lambda^2d^2-d^2}{d^2}=\lambda^2-1$.

3. Let $\frac{a_1}{b_1}=\lambda$, then $\frac{a_2}{b_2}=\lambda$. Then $a_1=\lambda b_1$, and $a_2=\lambda b_2$.

Then $\frac{m_1a_1+m_2a_2}{m_1b_1+m_2b_2}=\frac{m_1\lambda b_1+m_2\lambda b_2}{m_1b_1+m_2b_2}=\lambda=\frac{a_1}{b_1}$.

4. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{3a^2+ab+2b^2}{3a^2-2b^2}=\frac{3b^2\lambda^2+b^2\lambda+2b^2}{3b^2\lambda^2-2b^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}$$

$$\frac{3c^2+cd+2d^2}{3c^2-2d^2}=\frac{3d^2\lambda^2+d^2\lambda+2d^2}{3d^2\lambda^2-2d^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}.$$

5. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{a^2+3ab+b^2}{c^2+3cd+d^2}=\frac{b^2\lambda^2+3b^2\lambda+b^2}{d^2\lambda^2+3d^2\lambda+d^2}=\frac{b^2(\lambda^2+3\lambda+1)}{d^2(\lambda^2+3\lambda+1)}=\frac{b^2}{d^2}$$

$$\frac{2ab+3b^2}{2cd+3d^2}=\frac{2b^2\lambda+3b^2}{2d^2\lambda+3d^2}=\frac{b^2(2\lambda+3)}{d^2(2\lambda+3)}=\frac{b^2}{d^2}.$$

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}=\lambda$ and $\frac{e}{f}=\lambda$.

Then $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\frac{mc-ne}{md-nf}=\frac{md\lambda-nf\lambda}{md-nf}=\lambda=\frac{a}{b}.$

7. $$\frac{a-\frac{ma}{n}}{b-\frac{mb}{n}}=\frac{a\left(1-\frac{m}{n}\right)}{b\left(1-\frac{m}{n}\right)}=\frac{a}{b}.$$

8. Let $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\frac{ac}{bd}=\frac{bd\lambda^2}{bd}=\lambda^2$

and $\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}=\frac{lb^2\lambda^2+md^2\lambda^2+nf^2\lambda^2}{lb^2+md^2+nf^2}=\lambda^2.$

9. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_3\lambda$.

Then $\frac{a_1^2+a_2^2+a_3^2}{b_1^2+b_2^2+b_3^2}=\frac{b_1^2\lambda^2+b_2^2\lambda^2+b_3^2\lambda^2}{b_1^2+b_2^2+b_3^2}=\lambda^2=\frac{a_1^2}{b_1^2}.$

10. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_2\lambda$.

Then $\frac{a_1a_2+a_2a_3+a_3a_1}{b_1b_2+b_2b_3+b_3b_1}=\frac{b_1b_2\lambda^2+b_2b_3\lambda^2+b_3b_1\lambda^2}{b_1b_2+b_2b_3+b_3b_1}=\lambda^2=\frac{a_1^2}{b_1^2}.$

11. $(a^2-ab+b^2)(c^2+cd+d^2)=(a^2+ab+b^2)(c^2-cd+d^2)$;
$(a^2+b^2)(c^2+cd+d^2-c^2+cd-d^2)=ab(c^2-cd+d^2+c^2+cd+d^2)$;
$(a^2+b^2)(2cd)=ab(2c^2+2d^2)$; $a^2cd+b^2cd=abc^2+abd^2$;
$a^2cd-abd^2=abc^2-b^2cd$; $ad(ac-bd)=bc(ac-bd)$;
$ad=bc$; and therefore $a:b=c:d$.

12. $(a^2+b^2)(c^2-d^2)=(a^2-b^2)(c^2+d^2)$;
$a^2c^2+b^2c^2-a^2d^2-b^2d^2=a^2c^2+a^2d^2-b^2c^2-b^2d^2$; $2b^2c^2=2a^2d^2$;
$bc=ad$; and $\therefore$ $a:b=c:d$.

13. Let $\dfrac{a}{b}=\lambda$ and $\dfrac{c}{d}=\lambda$; then $a=b\lambda$ and $c=d\lambda$.

Then $\dfrac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\dfrac{(b\lambda+d\lambda)(b^2\lambda^2+d^2\lambda^2)}{(b\lambda-d\lambda)(b^2\lambda^2-d^2\lambda^2)}=\dfrac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}$.

14. Let $a_1=b_1\lambda$ and $a_2=b_2\lambda$.

Then $\dfrac{\sqrt{(a_1^2+a_2^2)}}{\sqrt{(b_1^2+b_2^2)}}=\sqrt{\dfrac{b_1^2\lambda^2+b_2^2\lambda^2}{b_1^2+b_2^2}}=\sqrt{(\lambda^2)}=\lambda=\dfrac{a_1}{b}$.

CXXXII.

1. $(a-b)c=(b-c)b$; $ac-bc=b^2-bc$; $ac=b^2$; $a:b=b:c$.

2. Let $a=\lambda b$ and $c=\lambda d$.

Then $\dfrac{(a^2+b^2)(a+b)}{a^3}=\dfrac{(\lambda^2b^2+b^2)(\lambda b+b)}{\lambda^3b^3}=\dfrac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$,

and $\dfrac{(c^2+d^2)(c+d)}{c^3}=\dfrac{(\lambda^2d^2+d^2)(\lambda d+d)}{\lambda^3d^3}=\dfrac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$.

Also, $\dfrac{\sqrt[4]{(ma^4+nc^4)}}{\sqrt[4]{(mb^4+nd^4)}}=\sqrt[4]{\dfrac{mb^4\lambda^4+nd^4\lambda^4}{mb^4+nd^4}}=\sqrt[4]{\lambda^4}=\lambda=\dfrac{a}{b}$.

3. Let $a=b\lambda$ and $c=d\lambda$.

Then $\dfrac{ma-nb}{ma+nb}=\dfrac{mb\lambda-nb}{mb\lambda+nb}=\dfrac{m\lambda-n}{m\lambda+n}$,

and $\dfrac{mc-nd}{mc+nd}=\dfrac{md\lambda-nd}{md\lambda+nd}=\dfrac{m\lambda-n}{m\lambda+n}$.

4. $(5a+3b)(7b+3c)=(7a+3b)(5b+3c)$;
$35ab+15ac+21b^2+9bc=35ab+21ac+15b^2+9bc$; $6b^2=6ac$;
$b^2=ac$; $a:b=b:c$.

5. Let $a:b=c:d$, and suppose a to be the greatest of the four.

Then $\because \frac{a}{b}=\frac{c}{d}$, and a is greater than b, c is greater than d,

and $\because \frac{a}{c}=\frac{b}{d}$, and a is greater than c, b is greater than d.

Again, if $a+b:m+n=m-n:a-b$,
and $a+b$ is greater than $m+n$
$m-n$ is greater than $a-b$
$\therefore a+b+m-n$ is greater than $m+n+a-b$
$\therefore 2b$ is greater than $2n$; and $\therefore b$ is greater than n.

6. $(x-1)(x+2)=(x-2)(2x+1)$; $x^2+x-2=2x^2-3x-2$; $x^2=4x$;
$x=4$ or 0.

7. $\frac{a}{b}+1=\frac{c}{d}+1$; $\frac{a}{b}=\frac{c}{d}$.

8. Suppose the bicycle went $5x$ and the tricycle $4x$ yards per minute. Then the tricycle had a start of $2x$ yards out of 1760 yards. Also the bicycle went 880 yards while the tricycle went
$1760-2x-176$. $\therefore 1760:1760-2x-176=5:4$;
$1760\times 4=(1584-2x)\times 5$; $10x=7920-7040$, etc.

9. $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$; $\therefore \frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$;
$\therefore \frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$; and since b^2 is greater than d^2,
$\therefore a^2-b^2$ is greater than c^2-d^2
$\therefore a^2+d^2$ is greater than b^2+c^2.

10. $(10a+b)(12c+d)=(10c+d)(12a+b)$;
$120ac+12bc+10ad+bd=120ac+12ad+10bc+bd$; $2bc=2ad$
$bc=ad$; $a:b=c:d$.

11. $2x=3y$, and $xy=600$.

Hence $x\times\frac{2x}{3}=600$; $x^2=900$; $x=30$, etc.

12. Let $\frac{a}{b}=\lambda$; then $\frac{b}{c}=\lambda$.

Then (1.) $\frac{a}{a+b}=\frac{\lambda b}{\lambda b+b}=\frac{\lambda}{\lambda+1}$

$$\frac{a-b}{a-c}=\frac{\lambda b-b}{\lambda b-c}=\frac{\lambda^2c-\lambda c}{\lambda^2c-c}=\frac{\lambda^2-\lambda}{\lambda^2-1}=\frac{\lambda}{\lambda+1}.$$

(2.) $(a^2+b^2)(b^2+c^2)=(\lambda^2b^2+b^2)(\lambda^2c^2+c^2)$

$$=\lambda^2c^2(\lambda^2+1)(\lambda^2c^2+c^2)=\lambda^2c^4(\lambda^2+1)^2$$

$$(ab+bc)^2=(\lambda b^2+\lambda c^2)^2=(\lambda^3c^2+\lambda c^2)^2=\lambda^2c^4(\lambda^2+1)^2.$$

13. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{a+b}{b}=\frac{b\lambda+b}{b}=\lambda+1$;

and $\frac{c+d}{d}=\frac{d\lambda+d}{d}=\lambda+1$.

Hence $\frac{ab-bc-dx+(bc+dx)}{bc+dx}=\frac{a-b-c+(b+c)}{b+c}$;

or, $\frac{ab}{bc+dx}=\frac{a}{b+c}$; $\frac{b}{bc+dx}=\frac{1}{b+c}$; $b^2+bc=bc+dx$, etc.

14. Let $a=\lambda b$, and $b=\lambda c$.

Then $\frac{a+mb}{a-mb}=\frac{\lambda b+mb}{\lambda b-mb}=\frac{\lambda+m}{\lambda-m}$;

and $\frac{b+mc}{b-mc}=\frac{\lambda c+mc}{\lambda c-mc}=\frac{\lambda+m}{\lambda-m}$.

15. $a=\frac{5b}{4}$; and therefore $\frac{a^2-b^2}{a^2+b^2}=\frac{\frac{25b^2}{16}-b^2}{\frac{25b^2}{16}+b^2}=\frac{9}{41}$.

16. Let the sides be $2\frac{1}{2}x$, $3\frac{3}{4}x$, and $4x$ yards.

Then $2\frac{1}{2}x+3\frac{3}{4}x+4x=205$; $\frac{41}{4}x=205$; $x=20$, etc.

17. Let the sides be $3x$, $4x$ and $5x$ yards.
Then $3x+4x+5x=480$; $12x=480$; $x=40$, etc.

18. Let $a+b$ be the greatest term of the proportion, then $a-b$ is the least.
Also, $\frac{a+b}{p+q}=\frac{p-q}{a-b}$.
Hence $\frac{(a+b)-(p+q)}{p+q}=\frac{(p-q)-(a-b)}{a-b}$.
Now $p+q$ is greater than $a-b$.
$\therefore$ $(a+b)-(p+q)$ is greater than $(p-q)-(a-b)$.
And $\therefore$ $(a+b)+(a-b)$ is greater than $(p+q)+(p-q)$.

19. Let x be the rate of the man's rowing, y the rate of the stream, in miles per hour.
Then $x+y=$ the man's rate down stream,
and $x-y=$ the man's rate up stream.
Hence $x+y:x-y=5:3$; $3x+3y=5x-5y$; $4y=x$.
Also $\frac{30}{x+y}+\frac{30}{x-y}=12$; $\frac{30}{5y}+\frac{30}{3y}=12$; $16=12y$; $y=1\frac{1}{3}$.

20. Let C contain x pints of brandy and y pints of water.
Then A contains $x+y$ pints of water, and B $x+y$ pints of brandy.
Hence when B and C are mixed, the mixture contains $2x+y$ pints of brandy ;
and when A and C are mixed, the mixture contains x pints of brandy ;
therefore $2x+y:x=9:1$; $2x+y=9x$; $y=7x$, etc.

21. Let x be the number of quarters ; y the price of each in shillings.
Then selling price : $xy=105:100$
$\therefore$ $\frac{105xy}{100}=xy+16\times 20$; $\therefore$ $xy=6400$.
Again $x(y+5)=xy+20y$; $\therefore$ $x=4y$.
Hence $4y^2=6400$; $y^2=1600$; $y=40$; $x=160$.

22. Let x be the price of the horse in pounds.
Then since cost price : gain $=100$: gain per cent.
$\therefore x : 144-x=100 : x$
$\therefore x^2+100x=14400$; $x^2+100x+2500=16900$; $x=80$.

23. Let x be the cost of the goods in pounds.
Then $x:96-x=100:x$; $x^2+100x=9600$; $x=60$.

24. Let x be the cost of the sheep in pounds.
Then $x:24-x=100:x$; $x^2+100x=2400$; $x=20$.

25. Let the first crew row x yards, and the second y yards at each stroke.
Then in 8 minutes the first crew row $320x$ yards,
and in 8 minutes the second crew row $360y$ yards.
Also the second crew has $4x$ yards start.
Hence $320x=360y+4x$; $316x=360y$; $79x=90y$, etc.

26. Let x be the rate of the fast train, y the rate of the slow train.
Then $\frac{180}{y}+\frac{30}{y}=$ time for journey by slow train
$\frac{180}{x}+\frac{15}{y}=$ time for journey by fast train
$\therefore \frac{210}{y} : \frac{180y+15x}{xy}=14:9$; or, $\frac{14}{y} : \frac{12y+x}{xy}=14:9$.
Hence $14\times 9\times x=(12y+x)\times 14$; $8x=12y$; $2x=3y$.
Again $x=y+15$; $\therefore 2y+30=3y$; $y=30$; $x=45$.

27. Let x be the worth of the article, y the selling price, in pounds.
Then $x:x-y=100:x$
$\therefore x^2=100x-100y$; $x^2-100x+2500=2500-100y$.
To obtain a real value for x, $100y$ must not be greater than 2500 ;
$\therefore y$ cannot be greater than 25.

CXXXIII.

1. Let $A=\frac{m}{B}$ and $B=\frac{n}{C}$.

Then $A=m\div\frac{n}{C}=\frac{m}{n}\cdot C$; $\therefore A \propto C$.

2. Let $A=mB$. Then $\frac{A}{P}=m\cdot\frac{B}{P}$; $\therefore \frac{A}{P} \propto \frac{B}{P}$.

3. Let $A=mB$ and $C=nD$. Then $AC=mn.BD$, etc.

4. $5:7=12:x$; $5x=84$, etc.

5. $x:\frac{1}{y}=10:\frac{1}{2}$; $\frac{x}{2}=\frac{10}{y}$; $\frac{4}{2}=\frac{10}{y}$; $y=5$.

6. $x:yz=1:2\times 3$; $6x=yz$; $6\times 4=y\times 2$; $y=12$.

7. $x:\frac{y}{z}=6:\frac{4}{3}$; $\frac{4x}{3}=\frac{6y}{z}$; $\frac{4x}{3}=\frac{6\times 5}{7}$; $x=3\frac{3}{14}$.

8. $3x+5y:5x+3y=31:25$; $75x+125y=155x+93y$; $32y=80x$, etc.

9. Let $A=mB$, and $B^3=nC^2$; then $\frac{A^3}{m^3}=nC^2$; $A^3=\frac{n}{m^3}\cdot C^2$;

$A=\frac{n^{\frac{1}{3}}}{m}\cdot C^{\frac{2}{3}}$; $\therefore A \propto C^{\frac{2}{3}}$.

10. $z:xy=4:2$; $30:3x=2:1$; $x=5$.

11. $A:B=8:12$; $12A=8B$; $A=\frac{2}{3}B$.

12. $x^2:y^3=9:64$; $64x^2=9y^3$.

13. $x^2:\frac{1}{y^3}=4:\frac{1}{27}$; $\frac{x^2}{27}=\frac{4}{y^3}$; $x^2=\frac{108}{y^3}$.

14. $x^3:y^2=27:4$; $4x^3=27y^2$.

15. Let $x=mz$, and $y=\frac{n}{z}$; then $xy=mn$; $x=mn.\frac{1}{y}$.

16. The area of a triangle $=\frac{1}{2}$(base × altitude).
Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the triangles;
then $a_1b_1=a_2b_2$, or, $a_1:a_2=b_2:b_1$.

17. The area of a parallelogram = base × altitude.

Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the parallelograms.

Then $a_1b_1=a_2b_2$; or, $a_1:a_2=b_2:b_1$.

18. Let $y=p+mx+nx^2$.

$$\text{Then}\ \left.\begin{aligned}6&=p+\ m+\ n\\11&=p+2m+4n\\18&=p+3m+9n\end{aligned}\right\};\quad\left.\begin{aligned}5&=m+3n\\7&=m+5n\end{aligned}\right\};\ n=1,\ m=2,\ p=3.$$

$\therefore\ y=3+2x+x^2.$

19. $10\times27:9\times9\times10=2\times27:3\times3\times$ required height in feet.

$$\text{Height}=\frac{9\times9\times10\times2\times27}{10\times27\times3\times3}\ \text{ft.}=18\ \text{ft.}$$

20. Let g, n, l, b represent the area of glass, the number, length, and breadth of the panes respectively.

Then suppose $g=m.nlb$, where m is a constant,

$n=\frac{p}{b^2}$, where p is a constant.

$l=\frac{q}{b}$, where q is a constant.

$$\text{Then}\qquad g=m.\frac{p}{b^2}.l.\frac{q}{l}=mpq.\frac{1}{b^2}=mpq.\frac{l^2}{q^2}=\frac{mp}{q}.l^2$$

$\therefore\ g$ varies as l^2.

CXXXIV.

1. $a=2,\ d=3,\ n-1=16$

$$z=2+16\times3=2+48=50.$$

2. $a=4,\ d=4,\ n-1=49$

$$z=4+4\times49=4+196=200.$$

3. $a=7,\ d=\frac{1}{4},\ n-1=15$

$$z=7+\frac{1}{4}\times15=7+\frac{15}{4}=10\frac{3}{4}.$$

4. $a=\frac{1}{2}, d=-\frac{3}{2}, n-1=22$

$$s=\frac{1}{2}-\frac{3}{2}\times 22=\frac{1}{2}-33=-32\frac{1}{2}.$$

5. $a=\frac{5}{6}, d=\frac{1}{2}-\frac{5}{6}=-\frac{1}{3}, n-1=11$

$$s=\frac{5}{6}-\frac{1}{3}\times 11=\frac{5}{6}-\frac{11}{3}=-2\frac{5}{6}.$$

6. $a=-12, d=4, n-1=13$

$$s=-12+4\times 13=-12+52=40.$$

7. $a=-3, d=8, n-1=15$

$$s=-3+8\times 15=-3+120=117.$$

8. $a=\frac{n-1}{n}, d=\frac{n-2}{n}-\frac{n-1}{n}=\frac{-1}{n}, n-1=n-1$

$$s=\frac{n-1}{n}-\frac{1}{n}\times (n-1)=\frac{n-1}{n}-\frac{n-1}{n}=0.$$

9. $a=x^2+2xy+y^2, d=-2xy, n-1=n-1$

$$s=x^2+2xy+y^2-2xy(n-1)=x^2+y^2-2(n-2)xy.$$

10. $a=\frac{a-b}{a+b}, d=\frac{3a-2b}{a+b}, n-1=n-1$

$$s=\frac{a-b}{a+b}+\frac{3an-2bn-3a+2b}{a+b}=\frac{3an-2bn-2a+b}{a+b}.$$

CXXXV.

1. $2a=4, d=1, n=100$

$$s=\frac{100}{2}\{2+99\times 1\}=50\times 101=5050.$$

2. $2a=4, d=2, n=50$

$$s=25\{4+49\times 2\}=25\times 102=2550.$$

3. $2a=6,\ d=4,\ n=20$

$$s=10\{6+19\times 4\}=10\times 82=820.$$

4. $2a=\frac{1}{2},\ d=\frac{1}{4},\ n=15$

$$s=\frac{15}{2}\left\{\frac{1}{2}+\frac{1}{4}\times 14\right\}=\frac{15}{2}\times 4=30.$$

5. $2a=-18,\ d=2,\ n=12$

$$s=6\{-18+2\times 11\}=6\times 4=24.$$

6. $2a=\frac{5}{3},\ d=-\frac{1}{3},\ n=17$

$$s=\frac{17}{2}\left\{\frac{5}{3}-\frac{1}{3}\times 16\right\}=\frac{17}{2}\times\left(-\frac{11}{3}\right)=-\frac{187}{6}=-31\frac{1}{6}.$$

7. $2a=2,\ d=1,\ n=n$

$$s=\frac{n}{2}\{2+1\times(n-1)\}=\frac{n}{2}.(n+1).$$

8. $2a=2,\ d=3,\ n=n$

$$s=\frac{n}{2}\{2+3(n-1)\}=\frac{n}{2}(3n-1)=\frac{3n^2-n}{2}.$$

9. $2a=2,\ d=7,\ n=n$

$$s=\frac{n}{2}\{2+7(n-1)\}=\frac{n}{2}(7n-5)=\frac{7n^2-5n}{2}.$$

10. $2a=\frac{2(n-1)}{n},\ d=-\frac{1}{n},\ n=n$

$$s=\frac{n}{2}\left\{\frac{2n-2}{n}-\frac{n-1}{n}\right\}=\frac{n}{2}\times\frac{n-1}{n}=\frac{n-1}{2}.$$

CXXXVI.

1. $-14=100+19d;\ 19d=-114;\ d=-6.$

2. $-x=x+50d;\ 50d=-2x;\ d=-\frac{x}{25}.$

3. $5\frac{1}{2}=-\frac{1}{2}+48d$; $48d=6$; $d=\frac{1}{8}$.

4. $-21\frac{3}{4}=-\frac{3}{4}+24d$; $24d=-21$; $d=-\frac{7}{8}$.

5. $-20=-10+5d$; $5d=-10$; $d=-2$.

6. $0=150+90d$; $90d=-150$; $d=-\frac{5}{3}$.

CXXXVII.

1. (1.) $$\left.\begin{array}{l}a+58d=70\\a+65d=84\end{array}\right\};\ 7d=14;\ d=2;\ a=70-116=-46.$$

(2.) $$\left.\begin{array}{l}a+19d=93-35b\\a+20d=98-37b\end{array}\right\};\ d=5-2b;\ a+95-38b=93-35b,\text{ etc.}$$

(3.) $$\left.\begin{array}{l}a+d=\frac{1}{2}\\a+54d=5\cdot8\end{array}\right\};\ \left.\begin{array}{r}54a+54d=27\\a+54d=5\cdot8\end{array}\right\};\ 53a=21\cdot2;\ a=\frac{2}{5}.$$

(4.) $$\left.\begin{array}{l}a+d=4\\a+86d=-30\end{array}\right\};\ \left.\begin{array}{r}86a+86d=344\\a+86d=-30\end{array}\right\};\ 85a=374;\ a=4\cdot4.$$

2. $$\left.\begin{array}{l}(a+2d)+(a+7d)=31\\(a+4d)+(a+9d)=43\end{array}\right\};\ \left.\begin{array}{l}2a+9d=31\\2a+13d=43\end{array}\right\};\ d=3,\ a=2.$$

Hence sum of 10 terms $=5\{4+27\}=155$.

3. $$\left.\begin{array}{l}a+(a+2d)=0\\(a+d)+(a+6d=40\end{array}\right\};\ \left.\begin{array}{l}2a+2d=0\\2a+7d=40\end{array}\right\};\ d=8,\ a=-8.$$

Hence sum of 7 terms$=\frac{7}{2}\cdot\{-16+48\}=\frac{7}{2}\times32=112$.

4. $$\left.\begin{array}{l}a+3d=24\\a+4d=33\end{array}\right\};\ d=9,\ a=-3.$$

Hence the 100th term is $-3+99\times9=-3+891=888$.

5. $302=5+(n-1)\times 3$; $300=3n$; $n=100$.

6. $s=\frac{20}{2}\left\{32\frac{1}{6}+19\times 32\frac{1}{6}\right\}=10\times 20\times\frac{193}{6}=6433\frac{1}{3}$.

7. From the formula $s=(a+z)\frac{n}{2}$

$s=(1+103)\times 26s.=2704s.=£135.4s.$

8. (1.) 41st term $=-5+9\times 40=355$

Sum $=(-5+355)\times\frac{41}{2}=7175$.

(2.) 41st term $=4a^2-4a^2\times 40=-156a^2$

Sum $=(4a^2-156a^2)\times\frac{41}{2}=-3116a^2$.

(3.) 41st term $=1+x+(4+2x)40=161+81x$

Sum $=(1+x+161+81x)\times\frac{41}{2}=3321+1681x$.

(4.) 41st term $=-4\frac{1}{2}+3\frac{1}{10}\times 40=119\frac{1}{2}$

Sum $=\left(-4\frac{1}{2}+119\frac{1}{2}\right)\times\frac{41}{2}=2357\frac{1}{2}$.

(5.) 41st term $=\frac{1}{4}+\frac{1}{5}\times 40=8\frac{1}{4}$

Sum $=\left(\frac{1}{4}+8\frac{1}{4}\right)\times\frac{41}{2}=174\frac{1}{4}$.

9. (1.) $a=1002$; $d=-8$; nth term $=a+(n-1)d$

$2=1002-8(n-1)$; $8n=1008$; $n=126$

$s=(10002+2)\times 63=63252$.

(2.) $186=-6+8(n-1)$; $8n=200$; $n=25$

$s=(-6+186)\times\frac{25}{2}=2250$.

(3.) $-72{\cdot}3x=2\frac{1}{2}x-1{\cdot}7x(n-1)$; $1{\cdot}7n=76{\cdot}5$; $n=45$

$s=(2{\cdot}5x-72{\cdot}3x)\times\frac{45}{2}=-1570{\cdot}5x.$

(4.) $-24=\frac{1}{2}-(n-1)\times\frac{1}{4}$; $\frac{n}{4}=24\frac{3}{4}$; $n=99$

$s=\left(\frac{1}{2}-24\right)\times\frac{99}{2}=-\frac{4653}{4}=-1163\frac{1}{4}.$

(5.) $139(1-m)=m-1+(n-1)2(1-m)$; $139=-1+2(n-1)$; $n=71$

$s=\{m-1+139(1-m)\}\times\frac{71}{2}=4899(1-m).$

(6.) $x-2=x+254-(n-1)\times 4$; $4n=2+258$; $n=65$

$s=\{x+254+x-2\}\frac{65}{2}=65x+8190.$

CXXXVIII.

1. $18=3+5d$; $d=3$; means are 6, 9, 12, 15.

2. $-2=2+6d$; $d=-\frac{2}{3}$; means are $1\frac{1}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-1\frac{1}{3}$.

3. $\frac{2}{3}=3+4d$; $d=-\frac{7}{12}$; means are $2\frac{5}{12}$, $1\frac{5}{6}$, $1\frac{1}{4}$.

4. $\frac{1}{3}=\frac{1}{2}+5d$; $d=-\frac{1}{30}$; means are $\frac{7}{15}$, $\frac{13}{30}$, $\frac{2}{5}$, $\frac{11}{30}$.

CXXXIX.

1. $n=m+4d$; $d=\frac{n-m}{4}$; means are $\frac{3m+n}{4}$, etc.

2. $m-1=m+1+5d$; $d=-\frac{2}{5}$; means are $\frac{5m+3}{5}$, etc.

3. $n^2+1=n^2+5d$; $d=\frac{1}{5}$; means are $\frac{5n^2+1}{5}$, etc.

4. $x^2-y^2=x^2+y^2+4d$; $d=-\frac{y^2}{2}$; means are $\frac{2x^2+y^2}{2}$, etc.

CXL.

1. $z=1\times 2^6=64$.

2. $z=4\times 3^9=4\times 19683=78732$.

3. $z=5\times 4^8=327680$.

4. $z=8\times\left(\frac{1}{2}\right)^{14}=\frac{2^3}{2^{14}}=\frac{1}{2^{11}}=\frac{1}{2048}$.

5. $z=2\times 3^8=13122$.

6. $z=\frac{1}{64}\times 4^{10}=\frac{4^{10}}{4^3}=4^7=16384$.

7. $z=-\frac{2}{3}\times\left(-\frac{1}{2}\right)^6=\frac{-2}{3\times 2^6}=\frac{-1}{3\times 2^5}=-\frac{1}{96}$.

CXLI.

1. $s=\frac{2(2^{15}-1)}{2-1}=2(32768-1)=65534$.

2. $s=\frac{1(3^6-1)}{3-1}=\frac{729-1}{2}=364$.

3. $s=\frac{a(x^{26}-1)}{x^2-1}$.

4. $s=\frac{a\left(\frac{1}{x^9}-1\right)}{\frac{1}{x}-1}=\frac{a}{x^8}\left(\frac{x^9-1}{x-1}\right)$.

5. $$s=\frac{(a^2-x^2)\left\{\frac{1}{(a+x)^7}-1\right\}}{\frac{1}{a+x}-1}=\frac{(a^2-x^2)}{(a+x)^6}\left\{\frac{1-(a+x)^7}{1-a-x}\right\}$$
$$=\frac{(a-x)\{1-(a+x)^7\}}{(a+x)^5.(1-a-x)}.$$

6. $s=\frac{2(3^n-1)}{3-1}=3^n-1$.

7. $s=\frac{7(2^n-1)}{2-1}=7(2^n-1)$.

8. $s=\frac{5\{(-2)^8-1\}}{-2-1}=\frac{5(1-256)}{3}=-425.$

9. $s=\frac{-\frac{2}{3}\left\{\left(-\frac{1}{2}\right)^7-1\right\}}{-\frac{1}{2}-1}=\frac{-\frac{2}{3}\left\{-\frac{1}{128}-1\right\}}{-\frac{3}{2}}=\frac{4}{9}\times\left(-\frac{129}{128}\right)$

$=-\frac{43}{96}.$

CXLII.

1. $s=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$

2. $s=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}.$

3. $s=\frac{3}{1-\frac{1}{9}}=\frac{3\times9}{8}=\frac{27}{8}.$

4. $s=\frac{\frac{2}{3}}{1-\frac{1}{2}}=\frac{4}{3}.$

5. $s=\frac{\frac{3}{4}}{1-\frac{1}{3}}=\frac{3\times3}{4\times2}=1\frac{1}{8}.$

6. $s=\frac{\frac{1}{2}}{1+\frac{2}{3}}=\frac{1\times3}{2\times5}=\frac{3}{10}.$

7. $s=\frac{8}{1-\frac{1}{12}}=\frac{8\times12}{11}=8\frac{8}{11}.$

8. $s=\frac{1{\cdot}5}{1-\frac{1}{3}}=\frac{1{\cdot}5\times3}{2}=2\frac{1}{4}.$

9. $s=\frac{64}{1-\frac{1}{4}}=\frac{64\times4}{3}=85\frac{1}{3}.$

10. $s=\frac{2x^3}{1+\frac{1}{8x^2}}=\frac{16x^5}{8x^2+1}.$

11. $s=\frac{a}{1-\frac{b}{a}}=\frac{a^2}{a-b}.$

12. $s=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{1}{9}.$

13. $s=\dfrac{x}{1+\dfrac{y}{x}}=\dfrac{x^2}{x+y}$.

14. $s=86\left\{\dfrac{\dfrac{1}{100}}{1-\dfrac{1}{100}}\right\}=86\times\dfrac{1}{99}=\dfrac{86}{99}$.

15. $s=\dfrac{5}{10}+\dfrac{4}{100}+\dfrac{4}{1000}+\ldots=\dfrac{5}{10}+\dfrac{\dfrac{4}{100}}{1-\dfrac{1}{10}}=\dfrac{5}{10}+\dfrac{4}{90}=\dfrac{49}{90}$.

16. $s=\dfrac{8}{10}+\dfrac{36}{1000}+\dfrac{36}{100000}+\ldots=\dfrac{8}{10}+\dfrac{\dfrac{36}{1000}}{1-\dfrac{1}{100}}=\dfrac{8}{10}+\dfrac{36}{990}=\dfrac{46}{55}$.

CXLIII.

1. $243=3f^4$; $f^4=81$; $f=3$, etc.
2. $1024=1\times f^5$; $f^5=4^5$; $f=4$, etc.
3. $16=1\times f^4$; $f^4=16$; $f=2$, etc.
4. $\dfrac{243}{64}=\dfrac{1}{2}\times f^5$; $f^5=\dfrac{243}{32}$; $f=\dfrac{3}{2}$, etc.

CXLIV.

1. (1.) $s=\{16+7(12-1)\}\times\dfrac{12}{2}=93\times 6=558$.

(2.) $s=\{232-8(10-1)\}\times\dfrac{10}{2}=160\times 5=800$.

(3.) $s=\dfrac{3}{1-\dfrac{1}{6}}=\dfrac{18}{5}$.

(4.) $s=\dfrac{2}{1+\dfrac{1}{8}}=\dfrac{16}{9}.$

(5.) $s=\left\{1-\dfrac{7}{6}(13-1)\right\}\times\dfrac{13}{2}=(-13)\times\dfrac{13}{2}=-\dfrac{169}{2}.$

(6.) $s=\dfrac{\dfrac{1}{2}\cdot\left\{\dfrac{2^6}{3^6}-1\right\}}{-\dfrac{2}{3}-1}=\dfrac{3}{10}\cdot\left(\dfrac{3^6-2^6}{3^6}\right)=\dfrac{665}{2430}=\dfrac{133}{486}.$

(7.) $s=\left\{1-\dfrac{3}{2}\times 28\right\}\times\dfrac{29}{2}=(-41)\times\dfrac{29}{2}=-\dfrac{1189}{2}.$

(8.) $s=\left\{\dfrac{10}{7}+\dfrac{2}{7}\times 7\right\}\times\dfrac{8}{2}=\dfrac{24}{7}\times 4=13\dfrac{5}{7}.$

(9.) $s=\dfrac{\dfrac{1}{3}}{1-\dfrac{2}{3}}=\dfrac{1}{3}\times\dfrac{3}{1}=1.$

(10.) $s=\left\{\dfrac{6}{5}-2\times 9\right\}\times\dfrac{10}{2}=\left(-\dfrac{84}{5}\right)\times 5=-84.$

(11.) The common factor $=-\sqrt{6}\div\dfrac{\sqrt{3}}{\sqrt{5}}=-\dfrac{\sqrt{30}}{\sqrt{3}}=-\sqrt{10}$

$$s=\frac{\sqrt{\dfrac{3}{5}}\{(-\sqrt{10})^8-1\}}{-\sqrt{10}-1}=\frac{\sqrt{3}.\{10000-1\}}{\sqrt{5}\{-\sqrt{10}-1\}}=-\frac{9999\sqrt{3}}{\sqrt{5}(\sqrt{10}+1)}.$$

(12.) $s=\dfrac{-\dfrac{7}{5}\left\{-\dfrac{5^5}{2^5}-1\right\}}{-\dfrac{5}{2}-1}=-\dfrac{2}{5}\left(\dfrac{5^5+2^5}{2^5}\right)=-\dfrac{3157}{80}.$

2. Let the series be a, af, af^2, af^3, af^4.
Then $a\times af\times af^2\times af^3\times af^4=32$; $a^5f^{10}=2^5$; $af^2=2$.

3. $b=\dfrac{a+c}{2}$ and $b'=\sqrt{(ac)}$

$\therefore\dfrac{b}{b'}=\dfrac{a+c}{2\sqrt{(ac)}}.$

4. The arithmetic mean is $\frac{a+b}{2}$; the geometric is $\sqrt{(ab)}$.

Now since the square of every number is *positive*

$(\sqrt{a}-\sqrt{b})^2$ is greater than 0

$a-2\sqrt{(ab)}+b$ is greater than 0.

$\therefore \frac{a+b}{2}$ is greater than $\sqrt{(ab)}$.

5. $a+(a+d)+(a+2d)=12$; $3a+3d=12$; $a+d=4$.
Also, $a+5d=12$. Hence $4d=8$; $d=2$; $a=2$.
Then sum of 6 terms $=\{4+10\}\times 3=42$.

6. Let f be the common factor: $af=b$ and $af^2=c$
$\therefore ac=a^2f^2=b^2$.

7. $2n\times\frac{1}{2n}=x^2$; $1=x^2$; $x=\pm 1$.

8. $2n+\frac{1}{2n}=2y$; $y=n+\frac{1}{4n}$.

9. The sum of the geometric progression is $\frac{3^4-1}{3-1}$ or $\frac{80}{2}$ or 40.

The sum of the arithmetic progression is $\{8+4(n-1)\}\frac{n}{2}$.

Hence $2n+2n^2=40$; $n^2+n=20$; whence $n=4$.

10. The first term is 1, the constant difference 1.

Hence $153=\{2+(7+n-1)\}\frac{7+n}{2}$; $306=(8+n)(7+n)$;

$n^2+15n=250$; $n=10$.

11. Let n be the number of terms.

Then $\{2+2(n-1)\}\frac{n}{2}=2n\times\frac{n}{2}=n^2$.

12. Let the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$.
Then $5a+10d=95$; and $\therefore a+2d=19$.

13. $22=\left\{6\frac{2}{3}+\frac{13}{9}(n-1)\right\}\frac{n}{2}$; $44=n\left(\frac{47+13n}{9}\right)$;
$396=47n+13n^2$; whence $n=4$.

14. Let $100x+10y+z$ be the number.
Then $x+z=2y$ (1)
and $\frac{100x+10y+z}{y+z}=107$; or $100x-97y-106z=0$ (2)
and $100x+10y+z-396=100z+10y+x$. (3)
From (3) $99x-396=99z$; or $x-z=4$.
From this and (1) $x=y+2$ and $z=y-2$.
Hence from (2) $100y+200-97y-106y+212=0$.
Thus $y=4$, $x=6$, $z=2$.

15. In any geometrical series a, af, etc.
the $(p+q)$th term is af^{p+q-1}
the $(p-q)$th term is af^{p-q-1}
the pth term is af^{p-1}.
Hence $mn=af^{p+q-1}\times af^{p-q-1}=a^2f^{2p-2}=(af^{p-1})^2$
$\therefore af^{p-1}=\sqrt{(mn)}$.

16. Let x and y be the numbers. Then
$$\left.\begin{array}{l}x-y=48\\ \frac{x+y}{2}=\sqrt{(xy)}+18\end{array}\right\};\quad \left.\begin{array}{l}x-y=48\\ x+y-2\sqrt{xy}=36\end{array}\right\};\quad \left.\begin{array}{r}x-y=48\\ \sqrt{x}-\sqrt{y}=6\end{array}\right\};$$
dividing, $\sqrt{x}+\sqrt{y}=8$; hence $2\sqrt{x}=14$; $\sqrt{x}=7$; $x=49$; $y=1$

17. $11=1+4d$; $4d=10$; $d=\frac{5}{2}$, etc.

18. $2{\cdot}748=\{{\cdot}068+(n-1)\times{\cdot}0004\}\times\frac{n}{2}$; $5{\cdot}496={\cdot}0676n+{\cdot}0004n^2$;
$54960=4n^2+676n$; $n^2+169n=13740$; $n=60$.

19. $-1=1+10d$; $10d=-2$; $d=-\frac{1}{5}$, etc.

20. The $(n+1)$th term is 2^n.

The sum of n terms is $\dfrac{2^n-1}{2-1}=2^n-1$.

21. Sum of the *first* p terms $=\dfrac{a(r^p-1)}{r-1}$.

Sum of the *second* p terms $=\dfrac{ar^p(r^p-1)}{r-1}$.

Sum of the *third* p terms $=\dfrac{ar^{2p}(r^p-1)}{r-1}$, and so on, the sums forming a geometric series whose common factor is r^p.

22.
$$\left.\begin{array}{l}(a-2d)+(a-d)+a+(a+d)+(a+2d)=55\\(a-2d)^2+(a-d)^2+a^2+(a+d)^2+(a+2d)^2=765\end{array}\right\};$$
$$\left.\begin{array}{l}5a=55\\5a^2+10d^2=765\end{array}\right\};\ a=11;\ d=4,\text{ etc.}$$

23. Let the series be a, af, af^2, af^3, af^4.
$$\left.\begin{array}{l}af^4-a:af^3-af=10:3\\af+af^3=2a^2f\end{array}\right\};\quad\left.\begin{array}{l}3f^4-3=10f^3-10f\\1+f^2=2a\end{array}\right\};$$
$$\left.\begin{array}{l}3(f^4-1)=10f(f^2-1)\\1+f^2=2a\end{array}\right\};\quad\left.\begin{array}{ll}3(f^2+1)=10f&(1)\\1+f^2=2a&(2)\end{array}\right\};$$
from (1) we find $f=3$; and then from (2) $a=5$, etc.

24. This is explained in Art. 472.

25. Let $100x+10y+z$ be the number. Then
$$\left.\begin{array}{l}y^2=xz\\x+y+z=13\\100x+10y+z+792=100z+10y+x\end{array}\right\}$$
From the third equation we get $x-z=-8$.

Hence $x^2-2xz+z^2=64$.

But $\quad 4xz \quad =4y^2$

$\therefore\ (x+z)^2=64+4y^2$.

Also $x+z=13-y$

$\therefore\ (13-y)^2=64+4y^2$.

Hence we get $y=3$; and then $x=1$ and $z=9$.

26. Let x be the increase per cent. Then the population of any year is $\frac{100+x}{100}$ times the population of the preceding year.

Hence $10000 \times \left(\frac{100+x}{100}\right)^4 = 14641.$

Take the fourth root of each side; then

$10 \times \frac{100+x}{100} = 11$; $100+x=110$; $x=10.$

CXLV.

1. First insert two arithmetic means between $\frac{1}{6}$ and $\frac{1}{24}$.

$\frac{1}{24} = \frac{1}{6} + 3d$; $3d = -\frac{3}{24}$; $d = -\frac{1}{24}$.

Hence the arithmetic means are $\frac{1}{8}$ and $\frac{1}{12}$

∴ the harmonic means are 8 and 12.

2. $\frac{1}{3} = \frac{1}{2} + 5d$; $5d = -\frac{1}{6}$; $d = -\frac{1}{30}$.

Hence the arithmetic means are $\frac{7}{15}, \frac{13}{30}, \frac{2}{5}, \frac{11}{30}$

∴ the harmonic means are $\frac{15}{7}, \frac{30}{13}, \frac{5}{2}, \frac{30}{11}$.

3. $\frac{2}{3} = 3 + 4d$; $4d = -\frac{7}{3}$; $d = -\frac{7}{12}$.

Hence the arithmetic means are $\frac{29}{12}, \frac{11}{6}, \frac{5}{4}$

∴ the harmonic means are $\frac{12}{29}, \frac{6}{11}, \frac{4}{5}$.

4. $18 = 3 + 5d$; $5d = 15$; $d = 3.$

Hence the arithmetic means are 6, 9, 12, 15

∴ the harmonic means are $\frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}$.

5. $2^{-1}=\frac{1}{2}$; and we have to insert five arithmetic means between -1 and 2.

$2=-1+6d$; $6d=3$; $d=\frac{1}{2}$.

Hence the arithmetic means are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$

$\therefore$ the harmonic means are -2, ∞, 2, 1, $\frac{2}{3}$.

6. $-2=2+6d$; $6d=-4$; $d=-\frac{2}{3}$.

Hence the arithmetic means are $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-\frac{4}{3}$

$\therefore$ the harmonic means are $\frac{3}{4}$, $\frac{3}{2}$, ∞, $-\frac{3}{2}$, $-\frac{3}{4}$.

7. $\frac{23}{6}=\frac{1}{3}+7d$; $7d=\frac{21}{6}$; $d=\frac{1}{2}$.

Hence the arithmetic means are $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, $\frac{7}{3}$, $\frac{17}{6}$, $\frac{10}{3}$

$\therefore$ the harmonic means are $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

8. $\frac{1}{3y}=\frac{1}{2x}+(n+1)d$; $d=\frac{2x-3y}{6xy(n+1)}$.

First arithmetic mean is $\frac{1}{2x}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+3y+2x-3y}{6xy(n+1)}$

$=\frac{3ny+2x}{6xy(n+1)}$.

Second arithmetic mean is $\frac{3ny+2x}{6xy(n+1)}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+4x-3y}{6xy(n+1)}$.

. . . .

nth arithmetic mean is $\frac{1}{3y}-\frac{2x-3y}{6xy(n+1)}=\frac{2nx+3y}{6xy(n+1)}$.

Hence the harmonic means are $\frac{6xy(n+1)}{3ny+2x}$, $\frac{6xy(n+1)}{3ny+4x-3y}$, . . . $\frac{6xy(n+1)}{2nx+3y}$.

9. Let $\frac{1}{x}$ and $\frac{1}{y}$ be the second and third terms.

Then $2, x, y$ are in arithmetic progression.

$$\left.\begin{aligned} &\text{Hence } 2+y=2x \\ &\text{and } \frac{1}{2}+\frac{1}{x}+\frac{1}{y}=\frac{11}{12} \end{aligned}\right\}; \quad \frac{1}{2}+\frac{1}{x}+\frac{1}{2x-2}=\frac{11}{12}$$

$$\frac{2x-2+x}{2x^2-2x}=\frac{5}{12}; \ 36x-24=10x^2-10x; \ 10x^2-46x=-24;$$

hence we find $x=4$ or $\frac{3}{5}$; and $\therefore y=6$ or $-\frac{4}{5}$.

The arithmetic progressions will therefore be

$$-4, \ -2, \ 0, \ 2, \ 4, \ 6, \text{ and } \frac{31}{5}, \ \frac{24}{5}, \ \frac{17}{5}, \ 2, \ \frac{3}{5}, \ -\frac{4}{5}.$$

$\therefore$ the harmonic progressions are

$$-\frac{1}{4}, \ -\frac{1}{2}, \ \infty, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{6} \text{ and } \frac{5}{31}, \ \frac{5}{24}, \ \frac{5}{17}, \ \frac{1}{2}, \ \frac{5}{3}, \ -\frac{5}{4}.$$

10. Let x and y be the numbers. Then

$$\frac{x+y}{2}=\sqrt{(xy)}+13 \ (1); \text{ and } \sqrt{(xy)}=\frac{2xy}{x+y}+12$$

$$\frac{2xy}{x+y}=\sqrt{(xy)}-12 \ (2).$$

Multiply (1) by (2); $xy=xy+\sqrt{(xy)}-156$.

Hence $\sqrt{(xy)}=156$; from this and (1) we find $x=104$, $y=234$.

11. $2b=a+c$ and $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$

$$\therefore 2b\left(\frac{1}{b}+\frac{1}{d}\right)=\frac{2}{c}(a+c); \ 2+\frac{2b}{d}=\frac{2a}{c}+2;$$

$\therefore bc=ad$, and $\therefore a:b=c:d$.

12. $$x=\frac{2mn}{m+n}; \ \frac{1}{x-m}+\frac{1}{x-n}=\frac{1}{\frac{2mn}{m+n}-m}+\frac{1}{\frac{2mn}{m+n}-n}$$

$$=\frac{m+n}{mn-m^2}+\frac{m+n}{mn-n^2}=\frac{m+n}{n-m}\left\{\frac{1}{m}-\frac{1}{n}\right\}=\frac{m+n}{mn}=\frac{1}{m}+\frac{1}{n}.$$

13. $$\left.\begin{aligned} x+y+z&=11\\ x^2+y^2+z^2&=49\\ y&=\frac{2xz}{x+z}\end{aligned}\right\};\quad \begin{aligned} &\left.\begin{aligned}(x+z)^2&=121-22y+y^2\\ x^2+z^2&=49-y^2\end{aligned}\right\}\\ &2xz=72-22y+2y^2\end{aligned}$$

Now $(x+z)y=2xz$.

$\therefore\ (11-y)y=72-22y+2y^2$; whence $y=3$.

Then $x+z=8$, and $xz=12$; whence $x=2$, $z=6$.

14. $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are the pth, qth, and rth terms of an arithmetic progression, suppose $a, a+d. \ . \ . \ .$

$$\text{Then }\left.\begin{aligned}\frac{1}{x}&=a+(p-1)d\\ \frac{1}{y}&=a+(q-1)d\\ \frac{1}{z}&=a+(r-1)d\end{aligned}\right\};\quad \begin{aligned}\frac{1}{x}-\frac{1}{z}&=(p-r)d\\ \frac{1}{z}-\frac{1}{y}&=(r-q)d\\ \frac{1}{y}-\frac{1}{x}&=(q-p)d\end{aligned}$$

$$\text{Hence }\left.\begin{aligned}z-x&=(p-r)xzd\\ y-z&=(r-q)yzd\\ x-y&=(q-p)xyd\end{aligned}\right\};$$

adding $0=(p-r)xzd+(r-q)yzd+(q-p)xyd$

$\therefore\quad 0=(p-r)xz+(r-q)yz+(q-p)xy.$

15. $\frac{2ab}{a+b}, \frac{2bc}{b+c}, \frac{2ca}{c+a}$ are in A. P.

$$\therefore\ \frac{bc}{b+c}-\frac{ab}{a+b}=\frac{ca}{c+a}-\frac{bc}{b+c}$$

$$\therefore\ \frac{b^2c-ab^2}{(b+c)(a+b)}=\frac{c^2a-bc^2}{(b+c)(c+a)},\text{ or, }\frac{b^2(c-a)}{a+b}=\frac{c^2(a-b)}{c+a}$$

$\therefore\ b^2(c^2-a^2)=c^2(a^2-b^2)$, or, $2b^2c^2=a^2b^2+a^2c^2$

$\therefore\ \frac{2}{a^2}=\frac{1}{c^2}+\frac{1}{b^2}$; $\because$ b^2, a^2, c^2 are in H. P.

Again, $\frac{a+b}{2ab}, \frac{b+c}{2bc}, \frac{c+a}{2ca}$ are in A. P.

$$\therefore\ \frac{b+c}{bc}-\frac{a+b}{ab}=\frac{c+a}{ac}-\frac{b+c}{bc}$$

$$\therefore\ \frac{1}{c}+\frac{1}{b}-\frac{1}{b}-\frac{1}{a}=\frac{1}{a}+\frac{1}{c}-\frac{1}{c}-\frac{1}{b}$$

$$\therefore\ \frac{1}{c}+\frac{1}{b}=\frac{2}{a}\ ;\ \therefore\ b,\ a,\ c \text{ are in H. P.}$$

16. (1.) When c is the arithmetic mean between a and b.

$$\frac{\frac{a+b}{2}+2a}{\frac{a+b}{2}-b}+\frac{\frac{a+b}{2}+2b}{\frac{a+b}{2}-a}=\frac{5a+b}{a-b}+\frac{a+5b}{b-a}$$

$$=\frac{5a+b-a-5b}{a-b}=\frac{4a-4b}{a-b}=4.$$

(2.) When c is the geometric mean between a and b.

$$\frac{\sqrt{(ab)}+2a}{\sqrt{(ab)}-b}+\frac{\sqrt{(ab)}+2b}{\sqrt{(ab)}-a}$$

$$=\frac{\sqrt{a}(\sqrt{b}+2\sqrt{a})}{\sqrt{b}(\sqrt{a}-\sqrt{b})}-\frac{\sqrt{b}(\sqrt{a}+2\sqrt{b})}{\sqrt{a}(\sqrt{a}-\sqrt{b})}=\frac{a\sqrt{b}+2a\sqrt{a}-b\sqrt{a}-2b\sqrt{b}}{\sqrt{(ab)}.(\sqrt{a}-\sqrt{b})}$$

$$=\frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}}$$; and since $a+b$ is greater than $2\sqrt{(ab)}$,

$$\therefore\ \frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}} \text{ is greater than } \frac{4\sqrt{(ab)}+3\sqrt{(ab)}}{\sqrt{(ab)}}, \text{ or, than } 7.$$

(3.) When c is the harmonic mean between a and b.

$$\frac{\frac{2ab}{a+b}+2a}{\frac{2ab}{a+b}-b}+\frac{\frac{2ab}{a+b}+2b}{\frac{2ab}{a+b}-a}=\frac{4ab+2a^2}{ab-b^2}+\frac{4ab+2b^2}{ab-a^2}$$

$$=\frac{4a^2b+2a^3-4ab^2-2b^3}{ab(a-b)}=\frac{2a^2+6ab+2b^2}{ab}.$$

Now a^2+b^2 is greater than $2ab$

$$\therefore\ \frac{2a^2+2b^2+6ab}{ab} \text{ is greater than } \frac{4ab+6ab}{ab}, \text{ or, than } 10.$$

CXLVI.

1. $n.(n-1)=12\times 11=132.$

2. $n.(n-1).(n-2)=16\times 15\times 14=3360.$

3. $n.(n-1).(n-2).(n-3)=20\times 19\times 18\times 17=116280.$

4. $n.(n-1).(n-2)(n-3)(n-4)=8\times 7\times 6\times 5\times 4=6720.$

5. Number of letters is 11, and three are twice repeated,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|2}\,.\,\underline{|2}\,.\,\underline{|2}}=4989600.$

6. $n.(n-1)\ .\ .\ .\ (n-7)=8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=40320.$

7. $n.(n-1)\ .\ .\ .\ (n-9)=\underline{|10}=3628800.$

8. (1.) The number of signals in which we have 3 flags of different colours is $5.4.3=60$.

(2.) When 2 flags in a signal are of the same colour, suppose red, we can put 4 different colours with these, and each of the resulting signals may be arranged in 3 different ways, that is, with both the red first, or both last, or one first and one last. Thus we shall have 12 different signals with two reds. Similarly we shall have 12 different signals with two of each of the other 4 colours.

$\therefore$ we shall have 5×12 or 60 in all.

(3.) When we have 3 of the same colour, we shall have 5 different signals in all.

$\therefore$ total number $=60+60+5=125.$

9. Number of permutations $=\dfrac{\underline{|7}}{\underline{|2}}=\dfrac{5040}{2}=2520.$

10. $n:n(n-1)(n-2)=1:20$;

$20n=n(n-1)(n-2)$; $n^2-3n+2=20$; $n=6.$

11. $m.(m-1)(m-2):(m+2)(m+1)m=1:5$;
$5m.(m-1)(m-2)=(m+2)(m+1)m$;
$5m^2-15m+10=m^2+3m+2$; $m=4$.

12. Number of letters is 7, and therefore the number of permutations in which *cd* stand first $=\underline{|5}=120$.

13. Number of letters is 9, one of which is repeated twice, one three times, and one four times.

$$\therefore \text{ Number of permutations } =\frac{\underline{|9}}{\underline{|2}\,.\,\underline{|3}\,.\,\underline{|4}}=9\times4\times7\times5=1260.$$

14. In *Conceit* we have 7 letters, one of which occurs twice,

$$\therefore \text{ number of permutations } =\frac{\underline{|7}}{\underline{|2}}=2520.$$

In *Talavera,* 8 letters, one of which is repeated three times,

$$\therefore \text{ number of permutations } =\frac{\underline{|8}}{\underline{|3}}=6720.$$

In *Calcutta,* 8 letters, 3 of them repeated twice,

$$\therefore \text{ number of permutations } =\frac{\underline{|8}}{2\times2\times2}=5040.$$

In *Proposition,* 11 letters, 2 repeated twice, 1 thrice,

$$\therefore \text{ number of permutations } =\frac{\underline{|11}}{\underline{|2}.\ \underline{|2}.\ \underline{|3}}=1663200.$$

In *Mississippi,* 11 letters, 2 repeated 4 times, 1 twice,

$$\therefore \text{ number of permutations } =\frac{\underline{|11}}{\underline{|4}.\ \underline{|4}.\ \underline{|2}}=34650.$$

CXLVII.

1. $${}_{100}C_4=\frac{100\times99\times98\times97}{1\times2\times3\times4}=3921225.$$

2. $${}_6C_5=\frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}=6.$$

3. Leave out a, and find how many combinations can be formed out of the remaining 9 letters, 4 at a time

${}_9C_4=\frac{9\times8\times7\times6\times5}{1\times2\times3\times4\times5}=126$, with each of which a will combine to form a word of 5 letters.

4. Out of the 19 consonants we get $\frac{19\times18\times17}{1\times2\times3}$ combinations of 3,

then $\frac{19\times18\times17}{1\times2\times3}\times5$ = number of ways in which we can get 3 consonants and 1 vowel, and each combination of 4 letters admits of $\underline{|4}$ permutations.

$\therefore$ number of words $=\frac{19.18.17}{1.2.3}\times5\times\underline{|4}=116280.$

5. $\frac{n.(n-1)(n-2)(n-3)}{1.2.3.4}:\frac{n.(n-1)}{1.2}=15:2$; $(n-2)(n-3)=90$;

$n=12.$

6. $\frac{n.(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\frac{18}{5}\times\frac{n.(n-1)(n-2)}{1.2.3}$;

$(n-3)(n-4)=18\times4$; $n=12.$

7. Number of words $=\frac{17.16.15}{1.2.3}\times\frac{5.4}{1.2}\times\underline{|5}=816000.$

8. Number $=\frac{12.11.10.9.8.7}{1.2.3.4.5.6}\times\frac{5.4.3}{1.2.3}\times\underline{|9}=3353011200.$

9. $n(n-1)(n-2)=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\times6$; $6(n-3)=24$;

$n-3=4$; $n=7.$

10. Taking the coins singly, by twos, by threes, and so on,

number of combinations $=6+\frac{6.5}{1.2}+\frac{6.5.4}{1.2.3}+\frac{6.5.4.3}{1.2.3.4}+\frac{6.5.4.3.2}{1.2.3.4.5}+1$

$=6+15+20+15+6+1=63.$

11. $\frac{n.(n-1)(n-2)}{1.2.3}=425\times n$; $(n-1)(n-2)=2550$; hence $n=52.$

12. Number $=\dfrac{12.11.10}{1.2.3}\times\dfrac{16.15.14}{1.2.3}=123200.$

13. (1.) The number is $\dfrac{36.35.34.33.32}{1.2.3.4.5}=376992.$

(2.) He will go out with as many different parties as can be formed by taking 4 out of 35.

$\therefore$ he goes with $\dfrac{35.34.33.32}{1.2.3.4}=52360.$

CXLVIII.

1. $a^4+4a^3x+\dfrac{4.3}{1.2}a^2x^2+\dfrac{4.3.2}{1.2.3}ax^3+x^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

2. $$b^6+6b^5c+\frac{6.5}{1.2}b^4c^2+\frac{6.5.4}{1.2.3}b^3c^3+\frac{6.5.4.3}{1.2.3.4}b^2c^4+\frac{6.5.4.3.2}{1.2.3.4.5}bc^5+c^6$$
$$=b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6.$$

3. $$a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\frac{7.6.5.4}{1.2.3.4}a^3b^4+\frac{7.6.5.4.3}{1.2.3.4.5}a^2b^5$$
$$+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}ab^6+b^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$$
$$+21a^2b^5+7ab^6+b^7.$$

4. $$x^8+8x^7y+\frac{8.7}{1.2}x^6y^2+\frac{8.7.6.}{1.2.3.}x^5y^3+\frac{8.7.6.5}{1.2.3.4}x^4y^4+\frac{8.7.6.5.4}{1.2.3.4.5}x^3y^5$$
$$+\frac{8.7.6.5.4.3}{1.2.3.4.5.6}x^2y^6+\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}xy^7+y^8$$
$$=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8.$$

5. $$5^4+4.5^3.4a+\frac{4.3}{1.2}.5^2.4^2a^2+\frac{4.3.2}{1.2.3}.5.4^3a^3+(4a)^4$$
$$=625+2000a+2400a^2+1280a^3+256a^4.$$

6. $$(a^2)^5+5(a^2)^4.bc+\frac{5.4}{1.2}(a^2)^3.(bc)^2+\frac{5.4.3}{1.2.3}(a^2)^2.(bc)^3+\frac{5.4.3.2}{1.2.3.4}a^2(bc)^4$$
$$+(bc)^5=a^{10}+5a^8bc+10a^6b^2c^2+10a^4b^3c^3+5a^2b^4c^4+b^5c^5.$$

CXLIX.

1. $a^6 - 6a^5x + \frac{6.5}{1.2}.a^4x^2 - \frac{6.5.4}{1.2.3}a^3x^3 + \frac{6.5.4.3}{1.2.3.4}a^2x^4 - \frac{6.5.4.3.2}{1.2.3.4.5}ax^5 + x^6$

$= a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$

2. $b^7 - 7b^6c + \frac{7.6}{1.2}b^5c^2 - \frac{7.6.5}{1.2.3}b^4c^3 + \frac{7.6.5.4}{1.2.3.4}.b^3c^4 - \frac{7.6.5.4.3}{1.2.3.4.5}b^2c^5$

$+ \frac{7.6.5.4.3.2}{1.2.3.4.5.6}.bc^6 - c^7$

$= b^7 - 7b^6c + 21b^5c^2 - 35b^4c^3 + 35b^3c^4 - 21b^2c^5 + 7bc^6 - c^7.$

3. $(2x - 3y)^5 = \left\{ 2x\left(1 - \frac{3y}{2x}\right) \right\}^5$

$= 32x^5 \left\{ 1 - 5.\frac{3y}{2x} + \frac{5.4}{1.2} \cdot \left(\frac{3y}{2x}\right)^2 - \frac{5.4.3}{1.2.3} \cdot \left(\frac{3y}{2x}\right)^3 \right.$

$\left. + \frac{5.4.3.2}{1.2.3.4} \cdot \left(\frac{3y}{2x}\right)^4 - \left(\frac{3y}{2x}\right)^5 \right\}$

$= 32x^5 \left\{ 1 - \frac{15}{2} \cdot \frac{y}{x} + \frac{5.2.9}{4} \cdot \frac{y^2}{x^2} - \frac{5.2.27}{8} \cdot \frac{y^3}{x^3} \right.$

$\left. + \frac{5.81}{16} \cdot \frac{y^4}{x^4} - \frac{243}{32} \cdot \frac{y^5}{x^5} \right\}$

$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.$

4. $1 - 5(2x) + \frac{5.4}{1.2}(2x)^2 - \frac{5.4.3}{1.2.3}(2x)^3 + \frac{5.4.3.2}{1.2.3.4}(2x)^4 - (2x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5.$

5. $1 - 10x + \frac{10.9}{1.2}x^2 - \frac{10.9.8}{1.2.3}x^3 + \frac{10.9.8.7}{1.2.3.4}.x^4 - \frac{10.9.8.7.6}{1.2.3.4.5}x^5$

$+ \frac{10.9.8.7}{1.2.3.4}x^6 - \frac{10.9.8}{1.2.3}x^7 + \frac{10.9}{1.2} \cdot x^8 - 10x^9 + x^{10}$

$= 1 - 10x + 45x^2 - 120x^3 + 210x^4 - 252x^5 + 210x^6 - 120x^7$

$+ 45x^8 - 10x^9 + x^{10}.$

6. $a^{24} - 8a^{21}b^2 + \frac{8.7}{1.2}a^{18}b^4 - \frac{8.7.6}{1.2.3}a^{15}b^6 + \frac{8.7.6.5}{1.2.3.4}a^{12}b^8 - \frac{8.7.6}{1.2.3}a^9b^{10}$

$+ \frac{8.7}{1.2}a^6b^{12} - 8a^3b^{14} + b^{16}$

$= a^{24} - 8a^{21}b^2 + 28a^{18}b^4 - 56a^{15}b^6 + 70a^{12}b^8 - 56a^9b^{10}$

$+ 28a^6b^{12} - 8a^3b^{14} + b^{16}$.

CL.

1. $\{(a+2b) - c\}^3 = (a+2b)^3 - 3(a+2b)^2.c + 3(a+2b).c^2 - c^3$

$= a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3$.

2. $\{(1-2x) + 3x^2\}^3 = (1-2x)^3 + 3(1-2x)^2.\ 3x^2 + 3(1-2x).9x^4 + 27x^6$

$= 1 - 6x + 12x^2 - 8x^3 + 9x^2 - 36x^3 + 36x^4 + 27x^4 - 54x^5 + 27x^6$

$= 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

3. $(x^3 - x^2 + x)^3 = x^3(x^2 - x + 1)^3 = x^3\{(1-x) + x^2\}^3$

$= x^3\{(1-x)^3 + 3(1-x)^2.x^2 + 3(1-x).x^4 + x^6\}$

$= x^3\{1 - 3x + 3x^2 - x^3 + 3x^2 - 6x^3 + 3x^4 + 3x^4 - 3x^5 + x^6\}$

$= x^9 - 3x^8 + 6x^7 - 7x^6 + 6x^5 - 3x^4 + x^3$.

4. $\{(1 + 2x^{\frac{1}{6}}) + 3x^{\frac{1}{3}}\}^3 = (1 + 2x^{\frac{1}{6}})^3 + 3(1 + 2x^{\frac{1}{6}})^2.3x^{\frac{1}{3}} + 3(1 + 2x^{\frac{1}{6}}).9x^{\frac{2}{3}} + 27x$

$= 1 + 6x^{\frac{1}{6}} + 12x^{\frac{1}{3}} + 8x^{\frac{1}{2}} + 9x^{\frac{1}{3}} + 36x^{\frac{1}{2}} + 36x^{\frac{2}{3}} + 27x^{\frac{2}{3}} + 54x^{\frac{5}{6}} + 27x$

$= 27x + 54x^{\frac{5}{6}} + 63x^{\frac{2}{3}} + 44x^{\frac{1}{2}} + 21x^{\frac{1}{3}} + 6x^{\frac{1}{6}} + 1$.

5. $\left\{ (x+1) - \frac{1}{x} \right\}^3 = (x+1)^3 - 3(x+1)^2.\frac{1}{x} + 3(x+1).\frac{1}{x^2} - \frac{1}{x^3}$

$= x^3 + 3x^2 + 3x + 1 - 3x - 6 - \frac{3}{x} + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$

$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$.

6. $\{(a^{\frac{1}{4}} + b^{\frac{1}{4}}) - c^{\frac{1}{4}}\}^3 = (a^{\frac{1}{4}} + b^{\frac{1}{4}})^3 - 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})^2.c^{\frac{1}{4}} + 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})c^{\frac{1}{2}} - c^{\frac{3}{4}}$

$= a^{\frac{3}{4}} + 3a^{\frac{1}{2}}b^{\frac{1}{4}} + 3a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}} - 3a^{\frac{1}{2}}c^{\frac{1}{4}} - 6a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}} - 3b^{\frac{1}{2}}c^{\frac{1}{4}} + 3a^{\frac{1}{4}}c^{\frac{1}{2}}$

$+ 3b^{\frac{1}{4}}c^{\frac{1}{2}} - c^{\frac{3}{4}}$.

CLI.

1. The 8th term of $(1+x)^{11}$ is $\frac{11.10.9.8.7.6.5}{1.2.3.4.5.6.7}\cdot x^7 = 330x^7$.

2. The 5th term of $(a^2-b^2)^{12}$ is $\frac{12.11.10.9}{1.2.3.4}\cdot(a^2)^{12-5+1}.(b^2)^4 = 495a^{16}.b$

3. The 4th term of $(a-b)^{100}$ is $-\frac{100.99.98}{1.2.3}\cdot a^{100-4+1}.b^3 = -161700a^{97}b$

4. The 9th term of $(2ab-cd)^{14}$ is $\frac{14.13.12.11.10.9.8.7}{1.2.3.4.5.6.7.8}\cdot(2ab)^{14-9+1}.(cd$

$= 192192a^6b^6c^8d^8$.

5. The 9th term of $(a-b)^{16}$ is $\frac{16.15.14.13.12.11.10.9}{1.2.3.4.5.6.7.8}\cdot a^{16-9+1}.b^8$

$= 12870a^8b^8$.

6. The 5th term of $(a^{\frac{1}{3}}+b^{\frac{1}{3}})^8$ is $\frac{8.7.6.5}{1.2.3.4}\cdot(a^{\frac{1}{3}})^{8-5+1}.(b^{\frac{1}{3}})^4 = 70a^{\frac{4}{3}}b^{\frac{4}{3}}$.

7. The 10th term of $(a-b)^{19}$ is $-\frac{19.18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8.9}\cdot a^{10}.b^9$

$= -92378a^{10}b^9$.

The 11th term of $(a-b)^{19}$ is therefore $92378a^9b^{10}$.

8. The 7th term of $(a+x)^{13}$ is $\frac{13.12.11.10.9.8}{1.2.3.4.5.6}\cdot a^7.x^6 = 1716a^7x^6$.

The 8th term of $(a+x)^{13}$ is therefore $1716a^6x^7$.

9. The coefficient of the rth term of $(a+x)^n$ is $\frac{n.(n-1)\ldots(n-r+2}{1.2\ .\ .\ .\ (r-1)}$

and the middle term of $(a+x)^{4n}$ is the $(2n+1)$th.

$$\therefore \text{ Coefficient } = \frac{4n.(4n-1)\ .\ .\ .\ (4n-2n-1+2)}{1.2\ .\ .\ .\ (2n+1-1)}$$

$$= \frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}\cdot\frac{2n.(2n-1)\ .\ .\ .\ 1}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-2)\ .\ .\ .\ 6.4.2}{2n.(2n-1)\ .\ .\ .\ 3.2.1}\cdot\frac{(4n-1)(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2.2\ .\ .\ .\ \text{to } 2n \text{ factors}.\ \frac{(4n-1).(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2^{2n}.\ \frac{1.3.5\ .\ .\ .\ (4n-1)}{1.2.3\ .\ .\ .\ 2n}.$$

10. The middle term is the $(2n+2)$th.

$$\therefore \text{ Coefficient } =\frac{(4n+2)(4n+1)\ .\ .\ .\ (4n+2-2n-2+2)}{1.2\ .\ .\ .\ (2n+2-1)}$$

$$=\frac{(4n+2).(4n+1)\ .\ .\ .\ (2n+2)}{1.2\ .\ .\ .\ (2n+1)}$$

$$=\frac{(4n+2)(4n)(4n-2)...(2n+4)(2n+2)}{(2n+1)(2n)(2n-1)...(n+1)}.\frac{(4n+1)(4n-1)...(2n+3)}{n.(n-1)...2.1}$$

$$=2.2\ .\ .\ .\ \text{to } (n+1) \text{ factors}.\ \frac{(4n+1)(4n-1)\ .\ .\ .\ (2n+3)}{n.(n-1)\ .\ .\ .\ 2.1}$$

$$=2^{n+1}.\frac{(2n+3)(2n+5)\ .\ .\ .\ (4n-1)(4n+1)}{1.2\ .\ .\ .\ n}.$$

CLII.

1. $$1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}x^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^4$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4.$$

2. $$1+\frac{2}{3}a+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}a^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}a^3$$

$$=1+\frac{2a}{3}-\frac{a^2}{9}+\frac{4a^3}{81}.$$

3. $$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\cdot\frac{x^3}{a^3}\right.$$

$$\left.+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1.2.3.4}\cdot\frac{x^4}{a^4}\right\}$$

$$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}.$$

4. $$1+x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}(2x)^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}(2x)^4$$

$$=1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}.$$

5. $$a^{\frac{3}{4}}\left\{1+\frac{x}{a}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{4x}{3a}\right)^2+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{4x}{3a}\right)^3\right\}$$

$$=a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}a^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}.x^3.$$

6. $$a^{\frac{1}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)}{1.2}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)\cdot\left(\frac{4}{5}-2\right)}{1.2.3}\cdot\frac{x^{\frac{3}{4}}}{a^{\frac{3}{4}}}\right\}$$

$$=a^{\frac{1}{5}}+\frac{4}{5}\cdot a^{-\frac{1}{20}}.x^{\frac{1}{4}}-\frac{2}{25}a^{-\frac{3}{10}}.x^{\frac{1}{2}}+\frac{4}{125}\cdot a^{-\frac{11}{20}}.x^{\frac{3}{4}}.$$

7. $$1-\frac{x^2}{2}+\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1.2}\cdot x^4-\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)}{\cdot 1.2.3}x^6$$
$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^8$$
$$=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}.$$

8. $$1-\frac{7a^2}{3}+\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)}{1.2}a^4-\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{1.2.3}a^6$$
$$=1-\frac{7a^2}{3}+\frac{14a^4}{9}-\frac{14a^6}{81}.$$

9. $$1-\frac{9x}{4}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}(3x)^2-\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot(3x)^3$$
$$=1-\frac{9x}{4}-\frac{27x^2}{32}-\frac{135x^3}{128}.$$

10. $$x^3\left\{1-\frac{3}{2}\cdot\frac{2y}{3x^2}+\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)}{1.2}\cdot\frac{4y^2}{9x^4}-\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1.2.3}\cdot\frac{8y^3}{27x^6}\right\}$$
$$=x^3-xy+\frac{y^2}{6x}+\frac{y^3}{54x^3}.$$

11. $$1-\frac{5}{6}x+\frac{\frac{5}{6}\cdot\left(\frac{5}{6}-1\right)}{1.2}x^2-\frac{\frac{5}{6}\left(\frac{5}{6}-1\right)\left(\frac{5}{6}-2\right)}{1.2.3}x^3$$
$$=1-\frac{5}{6}x-\frac{5}{72}x^2-\frac{35}{1296}x^3.$$

12. $$\left(\frac{2x}{3}\right)^{\frac{2}{3}}\left\{1-\frac{2}{3}\cdot\frac{9y}{4x}+\frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{1.2}\cdot\frac{81y^2}{16x^2}\right\}$$
$$=\left(\frac{2}{3}\right)^{\frac{2}{3}}.x^{\frac{2}{3}}-\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{1}{3}}y-\frac{3}{8}\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{4}{3}}y^2.$$

CLIII.

1. $$1-2a+\frac{(-2)(-2-1)}{1.2}a^2+\frac{(-2)(-2-1)(-2-2)}{1.2.3}a^3+$$
$$\frac{(-2)(-2-1)(-2-2)(-2-3)}{1.2.3.4}a^4=1-2a+3a^2-4a^3+5a^4.$$

2. $$1+3x+\frac{(-1)\cdot(-2)}{1.2}(3x)^2-\frac{(-1).(-2).(-3)}{1.2.3}(3x)^3+$$
$$\frac{(-1).(-2).(-3).(-4)}{1.2.3.4}(3x)^4=1+3x+9x^2+27x^3+81x^4.$$

3. $$1+x+\frac{(-4).(-5)}{1.2}\cdot\frac{x^2}{16}-\frac{(-4).(-5).(-6)}{1.2.3}\cdot\frac{x^3}{64}$$
$$=1+x+\frac{5x^2}{8}+\frac{5x^3}{16}.$$

4. $$1+x+\frac{(-2).(-3)}{1.2}\cdot\frac{x^2}{4}-\frac{(-2).(-3).(-4)}{1.2.3}\cdot\frac{x^3}{8}+$$
$$\frac{(-2).(-3).(-4).(-5)}{1.2.3.4}\cdot\frac{x^4}{16}=1+x+\frac{3x^2}{4}+\frac{x^3}{2}+\frac{5x^4}{16}.$$

5. $$a^{-10}-(-5)a^{-12}.2x+\frac{(-5)(-6)}{1.2}a^{-14}.4x^2-\frac{(-5).(-6)(-7)}{1.2.3}.a^{-16}.8x^3$$
$$+\frac{(-5).(-6).(-7).(-8)}{1.2.3.4}a^{-18}.16x^4$$
$$=a^{-10}+10a^{-12}x+60a^{-14}x^2+280a^{-16}.x^3+1120a^{-18}x^4.$$

6. $$a^{-2}-(-6)a^{-\frac{7}{3}}\cdot x^{\frac{1}{3}}+\frac{(-6)\cdot(-7)}{1.2}\cdot a^{-\frac{8}{3}}\cdot x^{\frac{2}{3}}-\frac{(-6)\cdot(-7)\cdot(-8)}{1.2.3}\cdot a^{-3}x$$
$$=\frac{1}{a^2}+\frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}}+\frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}}+\frac{56x}{a^3}.$$

CLIV.

1. $$1-\frac{1}{2}x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}x^4+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}+\frac{3x^4}{8}-\frac{5x^6}{16}+\frac{35x^8}{128}.$$

2. $$1+\frac{3}{2}x^2+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2}x^4-\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)\cdot\left(-\frac{9}{2}\right)}{1.2.3.4}x^8$$

$$=1+\frac{3x^2}{2}+\frac{15x^4}{8}+\frac{35x^6}{16}+\frac{315x^8}{128}.$$

3. $$x^{-2}\left\{1-\frac{2}{5}\cdot\frac{z^5}{x^5}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)}{1.2}\cdot\frac{z^{10}}{x^{10}}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)\cdot\left(-\frac{12}{5}\right)}{1.2.3}\frac{z^{15}}{x^{15}}\right\}$$

$$=x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{7}{25}x^{-12}z^{10}-\frac{28}{125}x^{-17}.z^{15}.$$

4. $$1-x+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}4x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}8x^3$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}16x^4$$

$$=1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{8}.$$

5. $$a^{-1}\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}\cdot\frac{x^4}{a^4}\right.$$

$$\left.+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}\cdot\frac{x^6}{a^6}\right\}$$

$$=\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}.$$

6. $$a^{-1}\left\{1-\frac{1}{3}\cdot\frac{x^3}{a^3}+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)}{1.2}\cdot\frac{x^6}{a^6}\right.$$
$$\left.+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)}{1.2.3}\cdot\frac{x^9}{a^9}\right\}=\frac{1}{a}-\frac{x^3}{3a^4}+\frac{2x^6}{9a^7}-\frac{14x^9}{81a^{10}}.$$

CLV.

The rth term of $(a+x)^n$ is $\frac{n.(n-1)\ .\ .\ .\ (n-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}.\ a^{n-r+1}$.

1. The rth term is $\frac{7\cdot 6\ .\ .\ .\ (9-r)}{1\cdot 2\ .\ .\ .\ (r-1)}x^{r-1}$.

2. The rth term is $(-1)^{r-1}\cdot\frac{12.11\ .\ .\ .\ (14-r)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}$.

3. The rth term is $(-1)^{r-1}\cdot\frac{8.7\ .\ .\ .\ (10-r)}{1.2\ .\ .\ .\ (r-1)}a^{9-r}\cdot x^{r-1}$.

4. The rth term is $\frac{9.8\ .\ .\ .\ (11-r)}{1.2\ .\ .\ .\ (r-1)}\cdot(5x)^{10-r}\cdot(2y)^{r-1}$.

5. The rth term is $\frac{(-2)\cdot(-3)\ .\ .\ .\ (-2-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}$.

$$=(-1)^{r-1}\cdot\frac{2.3\ .\ .\ .\ r}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}=(-1)^{r-1}\cdot r\cdot x^{r-1}.$$

6. The rth term is $\frac{4.5\ .\ .\ .\ (r+2)}{1.2\ .\ .\ .\ (r-1)}\cdot(3x)^{r-1}$

$$=\frac{r\cdot(r+1)\cdot(r+2)}{1.2.3}\cdot(3x)^{r-1}.$$

7. The rth term is $\frac{\frac{1}{2}\cdot\frac{3}{2}\ .\ .\ .\ \left(r-\frac{3}{2}\right)}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}$

$$=\frac{\left\{1.3.5\ \ldots\ (2r-3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ \ldots\ (r-1)}\cdot x^{r-1}$$

$$=\frac{1.3.5\ \ldots\ (2r-3)}{1.2\ \ldots\ (r-1)}\cdot\left(\frac{x}{2}\right)^{r-1}.$$

8. The rth term is $(-1)^{r-1}\cdot\dfrac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\ \ldots\ \left(r-\frac{7}{3}\right)}{1.2\ \ldots\ (r-1)}\cdot a^{\frac{1}{3}-r+1}x^{r-1}$

$$=(-1)^{r-1}\cdot\frac{\left\{1.2.5\ \ldots\ (3r-7)\right\}\times\left(\frac{1}{3}\right)^{r-1}}{1.2\ \ldots\ (r-1)}.a^{\frac{1}{3}}\cdot\left(\frac{x}{a}\right)^{r-1}$$

$$=(-1)^{r-1}\cdot\frac{1.2.5\ \ldots\ (3r-7)}{1.2\ \ldots\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(\frac{x}{3a}\right)^{r-1}$$

$$=\frac{1.2.5\ \ldots\ (3r-7)}{1.2\ \ldots\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(-\frac{x}{3a}\right)^{r-1}.$$

9. The rth term is $\dfrac{\frac{7}{2}\cdot\left(\frac{7}{2}+1\right)\ \ldots\ \left(\frac{7}{2}+r-2\right)}{1.2\ \ldots\ (r-1)}\cdot(2x)^{r-1}$

$$=\frac{\left\{7.9\ \ldots\ (2r+3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ \ldots\ (r-1)}\cdot(2x)^{r-1}$$

$$=\frac{7.9\ \ldots\ (2r+3)}{1.2\ \ldots\ (r-1)}\cdot x^{r-1}.$$

10. The rth term is $\dfrac{\frac{3}{4}\cdot\left(\frac{3}{4}+1\right)\ \ldots\ \left(\frac{3}{4}+r-2\right)}{1.2\ \ldots\ (r-1)}.(a^2)^{-\frac{3}{4}-r+1}.(x^2)^{r-1}$

$$=\frac{\{3.7\ \ldots\ (4r-5)\}\times\left(\frac{1}{4}\right)^{r-1}}{1.2\ \ldots\ (r-1)}.a^{-\frac{3}{2}}.\left(\frac{x^2}{a^2}\right)^{r-1}$$

$$=\frac{3.7\ \ldots\ (4r-5)}{1.2\ \ldots\ (r-1)}\cdot\frac{a^{-\frac{3}{2}}}{4^{r-1}}\cdot\left(\frac{x}{a}\right)^{2(r-1)}$$

11. The $(r+1)$th term is $\dfrac{3.4\ .\ .\ .\ (3+r-1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{3.4\ .\ .\ .\ (2+r)}{1.2\ .\ .\ .\ r}.x^r=\frac{(r+1)(r+2)}{2}.x^r.$$

12. The $(r+1)$th term is $\dfrac{\frac{1}{2}\left(\frac{1}{2}+1\right)\ .\ .\ .\ \left(\frac{1}{2}+r-1\right)}{1.2\ .\ .\ .\ r}(4x)^r$

$$=\frac{\{1.3\ .\ .\ .\ (2r-1)\}\times\left(\frac{1}{2}\right)^r}{1.2\ .\ .\ .\ r}.(4x)^r$$

$$=\frac{1.3\ .\ .\ .\ (2r-1)}{1.2\ .\ .\ .\ r}.(2x)^r.$$

13. The $(r+1)$th term is $\dfrac{2r.(2r-1)\ .\ .\ .\ (r+1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{2r.(2r-1)\ .\ .\ .\ (r+1).r.(r-1)\ .\ .\ .\ 2.1}{(\underline{|r})^2}.x^r$$

$$=\frac{(2r-1).(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2r.(2r-2)\ .\ .\ .\ 4.2}{\underline{|r}}.x^r$$

$$=\frac{(2r-1)(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2^r.(\underline{|r})}{\underline{|r}}.x^r$$

$$=\frac{1.3.5\ .\ .\ .\ (2r-1)}{1.2.3\ .\ .\ .\ r}.(2x)^r.$$

14. Coefficient of x^{r+1} in $(1+x)^{n+1}$ is $\dfrac{(n+1.)n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}$

Coefficient of x^r in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ r}$.

Coefficient of x^{r+1} in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}$.

Now $\dfrac{(n+1)n(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}=\dfrac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}.\dfrac{n+1}{r+1}$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}.\left(1+\frac{n-r}{r+1}\right)$$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}+\frac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}.$$

15. $$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}.(a)^{-\frac{1}{2}-3}.\left(\frac{1}{x}\right)^3=\frac{5}{16}a^{-\frac{7}{2}}.\frac{1}{x^3}.$$

16. $$\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2.3.4}.(a^2)^{\frac{3}{2}-4}.(-b^2)^4=\frac{3}{128}a^{-5}.b^8.$$

17. $$-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1.2.3.4.5.6.7.8}.(a^2)^{\frac{1}{2}-8}.(2x^2)^8$$
$$=-\frac{429}{128}.a^{-15}.x^{16}.$$

18. $$-\frac{m.(m+1)\ \ldots\ (m+8)}{1.2\ \ldots\ 9}.a^{-(m+9)}.b^9.$$

19. $$\frac{\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ \ldots\ \left(\frac{1}{m}-5\right)}{1.2\ \ldots\ 6}a^{\frac{1}{m}-6}b^6$$
$$=\frac{(1-m)(1-2m)\ \ldots\ (1-5m)}{1.2\ \ldots\ 6.m^6}.a^{\frac{1}{m}-6}.b^6.$$

CLVI.

1. $$\sqrt[3]{31}=\sqrt[3]{(27+4)}=3\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$$
$$=3\left\{1+\frac{1}{3}\cdot\frac{4}{27}+\frac{1}{2}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\frac{16}{729}\right.$$
$$+\frac{1}{6}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\frac{64}{19683}$$
$$\left.+\frac{1}{24}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\left(\frac{1}{3}-3\right)\cdot\frac{256}{531441}+\ldots\right\}$$
$$=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\frac{2560}{43046721}+\ldots$$
$$=3+\frac{6085724}{43046721}=3{\cdot}1413749\ \ldots$$

2 . $\sqrt[7]{108} = \sqrt[7]{(128-20)} = 2\left(1-\frac{5}{32}\right)^{7}$

$$=2\left\{1-\frac{1}{7}\cdot\frac{5}{32}+\frac{1}{2}\cdot\frac{1}{7}\left(\frac{1}{7}-1\right)\frac{25}{1024}\right.$$

$$-\frac{1}{6}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)\cdot\frac{125}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\cdot\left(\frac{1}{7}-2\right)\cdot\left(\frac{1}{7}-3\right)\cdot\frac{625}{1048576}-\ldots\right\}$$

$$=2\left\{1-\frac{5}{224}-\frac{75}{50176}-\frac{1625}{11239424}-\frac{40625}{2517630976}\ldots\right\}$$

$$=2\left\{1-\frac{56197120+3763200+364000+40625}{2517630976}\right\}$$

$$=2\left(1-\frac{60364945}{2517630976}\right)=\frac{2457266031}{1258815488}=1{\cdot}95204\ldots$$

3. $\sqrt[5]{260} = \sqrt[5]{(243+17)} = 3\left(1+\frac{17}{243}\right)^{\frac{1}{5}}$

$$=3\left\{1+\frac{1}{5}\cdot\frac{17}{243}+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{289}{59049}\right.$$

$$\left.+\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{4913}{14348907}\right\}$$

$$=3.\left\{1+\frac{17}{1215}-\frac{578}{1476225}+\frac{29478}{1793613375}\ldots\right\}$$

$$=3\left\{1+\frac{25095825-702270+29478}{1793613375}\ldots\right\}$$

$$=3.\left\{1+\frac{24423033}{1793613375}\right\}=3+\frac{24423033}{597871125}=3{\cdot}04084\ldots$$

4. $\sqrt[5]{31} = \sqrt[5]{(32-1)} = 2\left(1-\frac{1}{32}\right)^{\frac{1}{5}} = 2\left\{1-\frac{1}{5}\cdot\frac{1}{32}\right.$

$$+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{1}{1024}-\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{1}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\left(\frac{1}{5}-3\right)\cdot\frac{1}{1048576}\ldots\right\}$$

$$=2\left\{1-\frac{1}{160}-\frac{1}{12800}-\frac{3}{4096000}-\frac{21}{655360000}\cdots\right\}$$

$$=2\left\{1-\frac{4096000+51200+480+21}{655360000}\cdots\right\}$$

$$=2\left\{1-\frac{4147701}{655360000}\right\}=1{\cdot}98734\ldots$$

CLVII.

1. $\begin{array}{r} 23561\\ 42513\\ 645325\\ \hline 1045032 \end{array}$

2. $\begin{array}{r} 3074852\\ 4635628\\ 1247153\\ \hline 10070344 \end{array}$

3. $\begin{array}{r} 358423\\ 267862\\ \hline 80451 \end{array}$

4. $\begin{array}{r} 211010\\ 124321\\ \hline 31134 \end{array}$

5. $\begin{array}{r} 57264\\ 675\\ \hline 354604\\ 513354\\ 434070\\ \hline 51117344 \end{array}$

6. $\begin{array}{r} 1456\\ 6451\\ \hline 1456\\ 6523\\ 11312\\ 13101\\ \hline 14332216 \end{array}$

7. $\begin{array}{r} 5)243012\\ \hline 31450 \text{ rem. } 2. \end{array}$

8. $\begin{array}{r} 6)3756025\\ \hline 522256 \text{ rem. } 1. \end{array}$

9. $\begin{array}{ll} & 25|40|05|44(4112\\ & 24\\ \hline 121 & 140\\ & 121\\ \hline 1221 & 1505\\ & 1221\\ \hline 12222 & 24444\\ & 24444\\ \hline \end{array}$

10. $\begin{array}{ll} & 5|68|98|t1(2437\\ & 4\\ \hline 44 & 168\\ & 154\\ \hline 483 & 1498\\ & 1209\\ \hline 4867 & 28et1\\ & 28et1\\ \hline \end{array}$

CLVIII.

1. 7 | 1828
7 | 261 – 1
7 | 37 – 2
7 | 5 – 2
0 – 5

2. 6 | 1820
6 | 303 – 2
6 | 50 – 3
6 | 8 – 2
6 | 1 – 2
0 – 1

3. 12 | 43751
12 | 3645 – *e*
12 | 303 – 9
12 | 25 – 3
12 | 2 – 1
0 – 2

4. 5 | 3700
5 | 740 – 0
5 | 148 – 0
5 | 29 – 3
5 | 5 – 4
5 | 1 – 0
0 – 1

5. 2 | 7631
2 | 3815 – 1
2 | 1907 – 1
2 | 953 – 1
2 | 476 – 1
2 | 238 – 0
2 | 119 – 0
2 | 59 – 1
2 | 29 – 1
2 | 14 – 1
2 | 7 – 0
2 | 3 – 1
2 | 1 – 1
0 – 1

6. 12 | 215855
12 | 17987 – *e*
12 | 1498 – *e*
12 | 124 – *t*
12 | 10 – 4
0 – *t*

7. 7 | 790158
7 | 112879 – 5
7 | 16125 – 4
7 | 2303 – 4
7 | 329 – 0
7 | 47 – 0
7 | 6 – 5
0 – 6

8. 4 | 34002
4 | 4334 – 1
4 | 1043 – 2
4 | 122 – 0
4 | 14 – 1
4 | 2 – 1
0 – 2

9. $$\begin{array}{r|r} 12 & 8978 \\ \hline 12 & 816-2 \\ \hline 12 & 75-1 \\ \hline 12 & 6-t \\ \hline & 0-6 \end{array}$$

10. $$\begin{array}{r|r} 12 & 3256 \\ \hline 12 & 166-4 \\ \hline 12 & 11-1 \\ \hline & 0-8 \end{array}$$

11. $$\begin{array}{r|r} 8 & 37704 \\ \hline 8 & 4311-5 \\ \hline 8 & 480-1 \\ \hline 8 & 54-4 \\ \hline 8 & 6-1 \\ \hline & 0-6 \end{array}$$

12. $$\begin{array}{r|r} 4 & 5056 \\ \hline 4 & 1165-0 \\ \hline 4 & 214-3 \\ \hline 4 & 36-1 \\ \hline 4 & 6-3 \\ \hline 4 & 1-2 \\ \hline & 0-1 \end{array}$$

13. $$\begin{array}{r|r} 7 & 654321 \\ \hline 7 & e0738-5 \\ \hline 7 & 16e3e-3 \\ \hline 7 & 2858-3 \\ \hline 7 & 478-0 \\ \hline 7 & 7e-3 \\ \hline 7 & 11-4 \\ \hline 7 & 1-6 \\ \hline & 0-1 \end{array}$$

14. $$\begin{array}{r|r} 11 & 2304 \\ \hline 11 & 104-t \\ \hline 11 & 2-7 \\ \hline & 0-2 \end{array}$$

CLIX.

1. $$\left.\begin{array}{l} \frac{25}{36}\times 6=4+\frac{6}{36} \\ \frac{6}{36}\times 6=1+0 \end{array}\right\} \therefore \cdot 41 \text{ is the result.}$$

2. $$\frac{3}{11}\times 7=1+\frac{10}{11};\ \frac{10}{11}\times 7=6+\frac{4}{11};\ \frac{4}{11}\times 7=2+\frac{6}{11};$$
$$\frac{6}{11}\times 7=3+\frac{9}{11};\ \frac{9}{11}\times 7=5+\frac{8}{11};\ \frac{8}{11}\times 7=5+\frac{1}{11};$$

$$\frac{1}{11}\times 7=1+\frac{7}{11};\ \frac{7}{11}\times 7=4+\frac{5}{11};\ \frac{5}{11}\times 7=3+\frac{2}{11};$$

$$\frac{2}{11}\times 7=1+\frac{3}{11};\ \frac{3}{11}\times 7=1+\frac{10}{11}$$

$\therefore$ result is $\cdot\dot{1}6235504\dot{3}$.

3.
$$\begin{array}{r|l} 9 & 23 \\ \hline 9 & 2-5 \\ \hline & 0-2 \end{array} \qquad \begin{array}{r} \cdot125 \\ 9 \\ \hline 1\cdot125 \\ 9 \\ \hline 1\cdot125 \end{array}$$

$\therefore$ result is $25\cdot\dot{1}$.

4.
$$\begin{array}{r|l} 6 & 1820 \\ \hline 6 & 303-2 \\ \hline 6 & 50-3 \\ \hline 6 & 8-2 \\ \hline 6 & 1-2 \\ \hline & 0-1 \end{array} \qquad \begin{array}{r} \cdot3375 \\ 6 \\ \hline 2\cdot0250 \\ 6 \\ \hline 0\cdot1500 \\ 6 \\ \hline 0\cdot9000 \\ 6 \\ \hline 5\cdot4000 \\ 6 \\ \hline 2\cdot4000 \end{array}$$

$\therefore$ result is $12232\cdot2005\dot{2}$.

5. $2x^5+x^4+2x^3+5x^2+4x+2-17486=0$;

$(x-6)(2x^4+13x^3+80x^2+485x+2914)=0$;

$\therefore\ x=6$. Hence the scale is senary.

6. $x^6+7x^5+4x^4+6x^3+3x^2+5-511173=0$;

$(x-8)(x^5+15x^4+124x^3+998x^2+7987x+63896)=0$;

$\therefore\ x=8$. Hence the scale is octenary.

7. (1.) Let N be the number, and suppose

$N = a.10^n + b.10^{n-1} + \ . \ . \ . \ + m.100 + p.10 + q.$

Then $N = a(10^n - 1) + b(10^{n-1} - 1) + \ . \ . \ . \ + m(100 - 1)$
$+ p(10 - 1) + (a + b + \ . \ . \ . \ + m + p + q).$

Now all the expressions $10^n - 1, 10^{n-1} - 1 \ . \ . \ . \ 10^2 - 1, 10 - 1$ are divisible by $10 - 1$, or 9, and therefore by 3.

Hence N is divisible by 3, if $a + b + \ . \ . \ . \ + m + p + q$ be divisible by 3.

(2.) Let $N = a.10^n + b.10^{n-1} + \ . \ . \ . \ + m.100 + p.10 + q.$

Now 100 and all its multiples are divisible by 4,

$\therefore$ N is divisible by 4 if $10p + q$ be divisible by 4.

(3.) Let $N = a.10^n + b.10^{n-1} + \ . \ . \ . \ + m.100 + p.10 + q.$

Now 1000 and all its multiples are divisible by 8,

$\therefore$ N is divisible by 8 if $100m + 10p + q$ be divisible by 8.

(4.) Let $N = a.10^n + b.10^{n-1} + \ . \ . \ . \ + m.100 + p.10 + q.$

Now 10 and all its multiples are divisible by 5,

$\therefore$ N is divisible by 5 if $q = 5$ or $q = 0$.

(5.) Let N be the number, $p_n, p_{n-1}, \ . \ . \ . \ p_4, p_3, p_2, p_1$, the digits.

Then $N = p_1 + 10.p_2 + 100p_3 + 1000p_4 + \ . \ . \ . + 10^{n-1}.p_{n-1} + 10^n p_n$
$= p_1 - p_2 + p_3 - p_4 + \ . \ . \ . \ + (-1)^n.p_n$
$+ p_2(10 + 1) + p_3(10^2 - 1) + p_4(10^3 + 1) + \ . \ . \ . + p_n\{10^n - (-1)^n\}.$

Now $10 + 1$, $10^2 - 1$ . . . are all divisible by $10 + 1$, or 11,

$\therefore$ N is divisible by 11, if $(p_1 + p_3 + \ . \ . \ . \ ) - (p_2 + p_4 \ . \ . \ .)$ be divisible by 11.

8. Let $N = a.r^n + b.r^{n-1} + \ . \ . \ . \ + m.r^2 + pr + q,$

then $n = a + rb + \ . \ . \ . \ + m.r^{n-2} + p.r^{n-1} + q.r^n.$

Then $N - n = a(r^n - 1) + b.(r^{n-1} - r) + \ . \ . \ . \ - m(r^{n-2} - r^2)$
$- p(r^{n-1} - r) - q(r^n - 1)$, and each of the factors $r^n - 1$, $r^{n-1} - r \ . \ . \ . \ r^{n-2} - r^2$, $r^{n-1} - r$, $r^n - 1$ is divisible by $r - 1$;

$\therefore$ $N - n$ is divisible by $r - 1$.

CLX.

1. $$\begin{array}{r} \overline{3}\cdot1651553 \\ \overline{4}\cdot7505855 \\ 6\cdot6879746 \\ \overline{2}\cdot6150026 \\ \hline \overline{1}\cdot2187180 \end{array}$$

2. $$\begin{array}{r} \overline{4}\cdot6843785 \\ \overline{5}\cdot6650657 \\ 3\cdot8905196 \\ \overline{3}\cdot4675284 \\ \hline \overline{7}\cdot7074922 \end{array}$$

3. $$\begin{array}{r} 2\cdot5324716 \\ 3\cdot6650657 \\ \overline{5}\cdot8905196 \\ \cdot3156215 \\ \hline 2\cdot4036784 \end{array}$$

4. $$\begin{array}{r} 2\cdot483269 \\ \overline{3}\cdot742891 \\ \hline 4\cdot740378 \end{array}$$

5. $$\begin{array}{r} \overline{2}\cdot352678 \\ \overline{5}\cdot428619 \\ \hline 2\cdot924059 \end{array}$$

6. $$\begin{array}{r} \overline{5}\cdot349162 \\ \overline{3}\cdot624329 \\ \hline \overline{3}\cdot724833 \end{array}$$

7. $$\begin{array}{r} \overline{2}\cdot4596721 \\ 3 \\ \hline \overline{5}\cdot3790163 \end{array}$$

8. $$\begin{array}{r} \overline{7}\cdot429683 \\ 6 \\ \hline \overline{40}\cdot578098 \end{array}$$

9. $$\begin{array}{r} \overline{9}\cdot2843617 \\ 7 \\ \hline \overline{62}\cdot9905319 \end{array}$$

10. $$\begin{array}{r|r} 3 & \overline{6}\cdot3725409 \\ \cline{2-2} & \overline{2}\cdot1241803 \end{array}$$

11. $$\begin{array}{r|r} 6 & \overline{14}\cdot432962 \\ \cline{2-2} & \overline{3}\cdot738827 \end{array}$$

12. $$\begin{array}{r|r} 9 & \overline{4}\cdot53627188 \\ \cline{2-2} & \overline{1}\cdot61514132 \end{array}$$

CLXI.

1. $\text{Log } 128 = \log 2^7 = 7 \log 2 = 2\cdot1072100.$

$$\text{Log } 125 = \log \frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$$
$$= 3 - 3 \log 2 = 3 - \cdot9030900 = 2\cdot0969100.$$

$$\text{Log } 2500 = \log \frac{10000}{4} = \log 10000 - \log 4 = 4 - 2 \log 2$$
$$= 4 - \cdot6020600 = 3\cdot3979400.$$

2. $\text{Log } 50 = \log \dfrac{100}{2} = \log 100 - \log 2 = 2 - \cdot3010300 = 1\cdot6989700.$

$$\text{Log } \cdot005 = \log \frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \overline{3}\cdot6989700.$$

$$\text{Log } 196 = \log (49 \times 4) = 2 \log 7 + 2 \log 2 = 2\cdot2922560.$$

3. $\text{Log } 6 = \log 3 + \log 2 = \cdot 7781513.$

$\text{Log } 27 = 3 \log 3 = 1 \cdot 4313639.$

$\text{Log } 54 = \log (27 \times 2) = 3 \log 3 + \log 2 = 1 \cdot 7323939.$

$\text{Log } 576 = \log (9 \times 64) = 2 \log 3 + 6 \log 2 = 2 \cdot 7604226.$

4. $\text{Log } 60 = \log (2 \times 3 \times 10) = \log 2 + \log 3 + \log 10 = 1 \cdot 7781513.$

$\text{Log } \cdot 03 = \log \frac{3}{100} = \log 3 - 2 = \cdot 4771213 - 2 = \bar{2} \cdot 4771213.$

$\text{Log } 1 \cdot 05 = \log \frac{105}{100} = \log \frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = \cdot 0211893.$

$\text{Log } \cdot 0000432 = \log \frac{16 \times 27}{10000000} = 4 \log 2 + 3 \log 3 - 7 = \bar{5} \cdot 6354839.$

5. $\text{Log } \cdot 00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log \left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$

$= \frac{1}{2} \{\log 18 - \log 2\} + \log 100 - \log 4 - 5.$

$= \frac{1}{2} \{1 \cdot 2552725 - \cdot 3010300\} + 2 - \cdot 6020600 - 5$

$= \cdot 4771213 - \cdot 6020600 - 3 = \bar{4} \cdot 8750613.$

$\text{Log } 31 \cdot 5 = \log (21 \times 3 \times 5) - 1 = \log 21 + \log 3 + 1 - \log 2 - 1.$

$= \log 21 + \frac{1}{2} \{\log 18 - \log 2\} - \log 2$

$= 1 \cdot 3222193 + \cdot 4771212 - \cdot 3010300 = 1 \cdot 4983105.$

6. $\text{Log } 2 = \log \frac{10}{5} = 1 - \log 5 = \cdot 3010300$

$\text{Log } \cdot 064 = \log . \frac{2^6}{1000} = 6 \log 2 - 3 = 6 - 6 \log 5 - 3 = \bar{2} \cdot 8061800$

$\text{Log } \left\{ \frac{2^{60}}{5^{20}} \right\}^{\frac{1}{14}} = \frac{1}{14} (60 \log 2 - 20 \log 5)$

$= \frac{1}{7} (30 - 30 \log . 5 - 10 \log 5) = \frac{1}{7} (30 - 27 \cdot 9588000)$

$= \frac{1}{7} (2 \cdot 0412000) = \cdot 2916000.$

7. $\text{Log } 5=\log\frac{10}{2}=1-{\cdot}3010300={\cdot}6989700$

$$\log {\cdot}125=\log\frac{5^3}{1000}=3\log 5-3=2{\cdot}0969100-3=\bar{1}{\cdot}0969100$$

$$\text{Log}\left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}}=\log 5^{\frac{90}{15}}-\log 2^{\frac{40}{15}}=\log 5^6-\log 2^{\frac{8}{3}}$$

$$=6\log 5-\frac{8}{3}\log 2=6\ (\log 10-\log 2)-\frac{8}{3}\log 2$$

$$=4{\cdot}1938200-{\cdot}8027467=3{\cdot}3910733.$$

8. $-2, 0, 2\ ;\ 1, 0, -1.$

9. 1593 is greater than 10^3 and less than 10^4; characteristic 3; 1593 is greater than 12^2 and less than 12^3; characteristic 2.

10. $\frac{4^{3y}}{2^{4y}}=8\ ;\ \frac{2^{6y}}{2^{4y}}=2^3\ ;\ 2^{2y}=2^3\ ;\ 2y=3$, etc.

11. (*a*) $\log 2=\frac{1}{2}\log 4={\cdot}3010300.$

$$\text{Log } 25=\log 100-\log 4=2-{\cdot}6020600=1{\cdot}3979400$$

$$\text{Log } 83{\cdot}2=\log (80\times 1{\cdot}04)=\frac{3}{2}\log 4+\log 10+\log 1{\cdot}04$$

$$={\cdot}9030900+1+{\cdot}0170333=1{\cdot}9201233.$$

$$\text{Log } ({\cdot}625)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 625-\log 1000\right\}=\frac{1}{100}\left\{2\log 25-3\right\}$$

$$=\frac{1}{100}\left\{2\log 100-2\log 4-3\right\}=\frac{1}{100}\left\{4-1{\cdot}2041200-3\right\}$$

$$=-{\cdot}0020412=\bar{1}{\cdot}9979588.$$

(*b*) $\text{Log } (1{\cdot}04)^{6000}=6000\log 1{\cdot}04=6000\times{\cdot}0170333$

$=102{\cdot}1998000$; ∴ number of digits is 103.

12. (*a*) $\text{Log } 5=\frac{1}{2}\log 25=\cdot 6989700$

$\text{Log } 4=2-\log 25=\cdot 6020600$

$\text{Log } 51\cdot 5=\log 5+\log 10\cdot 3=\cdot 6989700+1\cdot 0128372=1\cdot 7118072$

$\text{Log } (\cdot 064)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 64-\log 1000\right\}=\frac{1}{100}\left\{3\log 4-3\right\}$

$=\frac{1}{100}\left\{1\cdot 8061800-3\right\}=-\cdot 0119382=\bar{1}\cdot 9880618.$

(*b*) $\text{Log } (1\cdot 03)^{600}=600\log 1\cdot 03=600\times\cdot 0128372$

$=7\cdot 7023200$; $\therefore$ number of digits is 8.

13. $\text{Log } 7623=2\log 3+2\log 11+\log 7$

$=\cdot 9542426+2\cdot 0827854+\cdot 8450980=3\cdot 8821260$

$\text{Log } \frac{77}{300}=\log 7+\log 11-\log 3-\log 100$

$=\cdot 8450980+1\cdot 0413927-\cdot 4771213-2=\bar{1}\cdot 4093694$

$\text{Log } \frac{3}{539}=\log 3-\log 11-2\log 7$

$=\cdot 4771213-1\cdot 0413927-1\cdot 6901960=\bar{3}\cdot 7455326.$

14. (1.) $x\log 4096=\log 8-x\log 64$; $4x\log 8=\log 8-2x\log 8$;

$4x=1-2x$; $6x=1$; $x=\frac{1}{6}$.

(2.) $(2\cdot 5)^x=6\cdot 25=(2\cdot 5)^2$; $\therefore x=2.$

(3.) $(ab)^x=m$; $x\log(ab)=\log m$

$\therefore x=\dfrac{\log m}{\log a+\log b}.$

(4.) $x(m\log a+2\log b)=\log c$, etc.

(5.) $3x\log a+(4-x)\log b=(2x-1)\log c$

$x(3\log a-\log b-2\log c)=-4\log b-\log c$, etc.

(6.) $x(\log a+m\log b)=\log c-3x\log c$

$x(\log a+m\log b+3\log c)=\log c$, etc.

CLXII.

1. $P(1+r)^n=2P$; $(1+r)^n=2$; $\left(1+\frac{4}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 104-\log 100}=\frac{\cdot 3010300}{\cdot 0170333}=17\cdot 6\ .\ .\ .$$

2. $P(1+r)^n=2P$; $\left(1+\frac{3}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 4\ .\ .\ .$$

3. $\left(1+\frac{10}{100}\right)^n=2$; $n=\frac{\log 2}{\log 1\cdot 1}=\frac{\cdot 3010300}{\cdot 0413927}=7\cdot 2725\ .\ .\ .$

4. $\left(1+\frac{5}{100}\right)^n=3$; $n=\frac{\log 3}{\log 1\cdot 05}=\frac{\cdot 4771213}{\cdot 0211893}=22\cdot 5$ nearly.

5. $P(1+r)^n=2P$; $\therefore n=\frac{\log 2}{\log (1+r)}$

$$P(1+2r)^m=2P\ ;\ \therefore m=\frac{\log 2}{\log (1+2r)}$$

$$\therefore \frac{m}{n}=\frac{\log (1+r)}{\log (1+2r)}.$$

Now $1+2r$ is less than $(1+r)^2$

$\therefore \frac{m}{n}$ is greater than $\frac{\log (1+r)}{\log (1+r)^2}$, or, $\frac{\log (1+r)}{2\log (1+r)}$, or, $\frac{1}{2}$.

6. $1000(1+r)^n=1800$; $\left(1+\frac{5}{100}\right)^n=1\cdot 8$

$\therefore n \log 1\cdot 05=\log 1\cdot 8$; $n=\frac{\cdot 2552725}{\cdot 0211893}=12$ nearly.

7. $P(1+r)^{2n}=2P$; $\left(1+\frac{3}{100}\right)^{2n}=2$

$$\therefore 2n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 449\ .\ .\ .$$

$$\therefore n=11\cdot 724\ .\ .\ .\ .$$

FINIS.

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these too far apart, and the intercourse of the defenders with an army of relief under the Count of Clermont at Blois was not broken off. Early in the following year, this army hoped to raise the siege by falling on a large body of provisions coming to the besiegers from Paris under Sir John Fastolf. The attack was made at Rouvray, but Fastolf had made careful preparations. The waggons were arranged in a square, and, with the stakes of the archers, formed a fortification on which the disorderly attack of the French made but little impression. Broken in the assault, they fell an easy prey to the English, as they advanced beyond their lines. The skirmish is known by the name of the Battle of the Herrings. This victory, which deprived the besieged of hope of external succour, seemed to render the capture of the city certain.

Battle of the Herrings.

Already at the French King's court at Chinon there was talk of a hasty withdrawal to Dauphiné, Spain, or even Scotland; when suddenly there arose one of those strange effects of enthusiasm which sometimes set all calculation at defiance.

Danger of Orleans.

In Domrémi, a village belonging to the duchy of Bar, the inhabitants of which, though in the midst of Lorraine, a province under Burgundian influence, were of patriotic views, lived a village maiden called Joan of Arc. The period was one of great mental excitement; as in other times of wide prevailing misery, prophecies and mystical preachings were current. Joan of Arc's mind was particularly susceptible to such influences, and from the time she was thirteen years old, she had fancied that she heard voices, and had even seen forms, sometimes of the Archangel Michael, sometimes of St. Catherine and St. Margaret, who called her to the assistance of the Dauphin. She persuaded herself that she was destined to fulfil an old prophecy which said that the kingdom, destroyed by a woman—meaning, as she thought, Queen Isabella,—should be saved by a maiden of Lorraine. The burning of Domrémi in the summer of 1428 by a troop of Burgundians at length gave a practical form to her imaginations, and early in the following year she succeeded in persuading Robert of Baudricourt to send her, armed and accompanied by a herald, to Chinon. She there, as it is said by the wonderful knowledge she displayed, convinced the court of the truth of her mission. At all events, it was thought wise to take advantage of the infectious enthusiasm she displayed, and in April she was intrusted with an army of 6000 or 7000 men, which was to march up the river from Blois to the relief of Orleans. When she appeared upon the scene of war, she supplied exactly that element of success

Joan of Arc.

of all of them open by two slits turned towards the centre of the flower. Their stalks have expanded and joined together, so as to form a thin sheath round the central column (fig. 12). The dust-spikes are so variable in length in this flower, that it may not be possible to see that one short one comes between two long ones, though this ought to be the case.

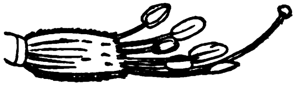

Fig. 12.
Dust-spikes of gorse (*enlarged*).

The *seed-organ* is in the form of a longish rounded pod, with a curved neck, stretching out beyond the dust-spikes. The top of it is sticky, and if you look at a bush of gorse, you will see it projecting beyond the keel in most of the fully-blown flowers, because the neck has become more curved than in fig. 12. Cut open the pod; it contains only one cavity (not, as that of the wall-flower, two separated by a thin partition), and the grains are suspended by short cords from the top (fig. 13). These grains may be plainly seen in the seed-organ of even a young flower. It is evident that they are the most important part of the plant, as upon them depends its diffusion and multiplication. We have already seen how carefully their well-being is considered in the matter of their perfection, how even insects are pressed into their service for this purpose! Now let us glance again at our flower, and see how wonderfully contrivance is heaped upon contrivance for their protection!

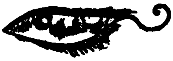

Fig. 13.
Split seed-pod of gorse.

First (see fig. 10, p. 14), we have the outer covering, so covered with hairs, that it is as good for keeping out rain as a waterproof cloak; in the buttercup, when you pressed the bud, it separated into five leaves; here there are five leaves, just the same, but they are so tightly joined that you may press till the whole bud is bent without making them separate at all, and when the bud is older, they only separate into two, and continue to enfold the flower to a certain extent till it fades. When the flower pushes back its waterproof cloak, it has the additional shelter of the big struction, and at last, after nearly twenty years of alternate hopes and fears, of tedious negotiations, official evasions, and sterile Parliamentary debates, it was effectually extinguished by the adverse report of a Parliamentary Committee, followed by the erection of the present Millbank Penitentiary at a vastly greater expense and on a totally different system.

Transportation.—In the meantime the common gaols were relieved in a makeshift fashion by working gangs of prisoners in hulks at the seaports; but the resource mainly relied on for getting rid of more dangerous criminals was the old one of transportation, Botany Bay having succeeded to America. As at first employed, there was no mistake as to the reality of the punishment; the misfortune was that the worst elements in the real were not so made known as to form any part of the apparent punishment. If the judge, in sentencing the convict, had thought fit to explain, for the warning of would-be offenders, exactly what was going to be done with their associate, the sentence would have been something of this sort: "You shall first be kept, for days or months as it may happen, in a common gaol, or in the hulks, in company with other criminals better or worse than yourself, with nothing to do, and every facility for mutual instruction in wickedness. You shall then be taken on board ship with similar associates of both sexes, crammed down between decks, under such circumstances that about one in ten of you will probably die in the course of the six months' voyage. If you survive the voyage you will either be employed as a slave in some public works, or let out as a slave to some of the few free settlers whom we have induced to go out there. In either case you will be under very little regular inspection, and will have every opportunity of indulging those natural wealth into the treasury. Churches remained open day and night, and frequent addresses kept up the enthusiasm to a high pitch. It was (for the moment) a genuine "revival" or reawakening of the whole Roman world. The occasion, too, appeared favourable. Italy was quiet, and the Exarchate at peace with its neighbours. Clotaire the Frank was no enemy to Heraclius, and in common with his clergy (being orthodox and not Arian) might be expected to sympathise in so holy a cause.

Treachery of the Avars—A.D. 616.—In one quarter only was there room for fear. The Avars were on the Danube, and the turbulence of the Avars was only equalled by their perfidy. Already, in A.D. 610, they had fallen suddenly on North Italy, and pillaged and harassed those same Lombards whom they had before helped to destroy the Gepidæ. Previous to an absence, therefore, of years from his capital, it was essential for the Emperor to sound their intentions, and, if possible, to secure their neutrality. His ambassadors were welcomed with apparent cordiality, and an interview was arranged between the Chagan and Heraclius. The place was to be Heraclea. At the appointed time the Emperor set out from Selymbria to meet the Khan, decked with Imperial crown and mantle to honour the occasion. The escort was a handful of soldiers; but there was an immense cortége of high officials and of the fashionable world of Constantinople, and the whole country side was there to see. Presently some terrified peasants were seen making their way hurriedly towards Heraclius. They urged him to flee for his life; for armed Avars had been seen in small bodies, and might even now be between him and the capital. Heraclius knew too much to hesitate. He threw off his robes and fled, and but just in time. The Chagan had laid a deep plot. A large mass of men had been told off in small detachments

I say the pulpit (in the sober use
Of its legitimate peculiar pow'rs)
Must stand acknowledg'd, while the world shall stand,
The most important and effectual guard,
Support and ornament of virtue's cause.
There stands the messenger of truth: there stands
The legate of the skies; his theme divine,
His office sacred, his credentials clear.
By him, the violated law speaks out
Its thunders, and by him, in strains as sweet
As angels use, the Gospel whispers peace.
He stablishes the strong, restores the weak,
Reclaims the wand'rer, binds the broken heart,
And, arm'd himself in panoply complete
Of heav'nly temper, furnishes with arms
Bright as his own, and trains, by ev'ry rule
Of holy discipline, to glorious war,
The sacramental host of God's elect.
Are all such teachers? would to heav'n all were!
But hark—the Doctor's voice—fast wedged between
Two empirics he stands, and with swoln cheeks
Inspires the news, his trumpet. Keener far
Than all invective is his bold harangue,
While through that public organ of report
He hails the clergy; and, defying shame,
Announces to the world his own and theirs.
He teaches those to read, whom schools dismiss'd,
And colleges, untaught; sells accent, tone,
And emphasis in score, and gives to pray'r
Th' *adagio* and *andante* it demands.
He grinds divinity of other days
Down into modern use; transforms old print
To zigzag manuscript, and cheats the eyes
Of gall'ry critics by a thousand arts.—
Are there who purchase of the Doctor's ware?
Oh name it not in Gath!—it cannot be,
That grave and learned Clerks should need such aid.
He doubtless is in sport, and does but droll,
Assuming thus a rank unknown before,
Grand caterer and dry-nurse of the church.

I venerate the man whose heart is warm,
Whose hands are pure, whose doctrine and whose life.

[COWPER'S TASK—*See Page* 6.]

[*Specimen Page, No.* 6.]

gether as with a close seal. . . . . The flakes of his flesh are joined together: they are firm in themselves; they cannot be moved."

Hobbes, in his famous book to which he gave the title *Leviathan*, symbolised thereby the force of civil society, which he made the foundation of all right.

315–325 Cowper's limitation of the province of satire—that it is fitted to laugh at foibles, not to subdue vices—is on the whole well-founded. But we cannot forget Juvenal's famous "facit indignatio versum," or Pope's no less famous—

> "Yes, I am proud: I must be proud to see
> Men not afraid of God, afraid of me:
> Safe from the bar, the pulpit, and the throne,
> Yet touched and shamed by ridicule alone."

326–372 *The pulpit, not satire, is the proper corrector of sin. A description of the true preacher and his office, followed by one of the false preacher, "the reverend advertiser of engraved sermons."*

330 *Strutting and vapouring.* Cf. *Macbeth*, v. 5.

> "Life 's but a walking shadow, a poor player,
> That struts and frets his hour upon the stage,
> And then is heard no more; it is a tale
> Told by an idiot, full of sound and fury,
> Signifying nothing."
>
> "And what in real value 's wanting,
> Supply with vapouring and ranting."—HUDIBRAS.

331 *Proselyte.* προσήλυτος, a new comer, a convert to Judaism.

338 *His theme divine.* Nominative absolute.

343 *Stablishes.* Notice the complete revolution the word has made—stabilire, établir, establish, stablish; cf. state, &c.

346 *Of heavenly temper.* Cf. *Par. Lost*, i. 284, "his ponderous shield etherial temper." See note on *Winter Morning Walk*, l. 664.

349 *Sacramental.* Used in the Latin sense. Sacramentum was the oath of allegiance of a Roman soldier. The word in its Christian sense was first applied to baptism—the vow to serve faithfully under the banner of the cross. See *Browne on the Thirty-nine Articles*, p. 576.

350 *Would to heaven.* A confusion between "would God" and "I pray to heaven."

351 A picture from the life of a certain Dr Trusler, who seems to have combined the trades of preacher, teacher of elocution, writer of sermons, and literary hack.

352 *Empirics.* ἐμπειρικός, one who trusts solely to experience or practice instead of rule, hence a quack. The accent is the same as in Milton (an exception to the rule. See note on *Sofa*, l. 52).

[COWPER'S TASK—*See Page* 6.]

thus: if the articles had cost £1 each, the total cost would have been £2478;

∴ as they cost $\frac{1}{6}$ of £1 each. the cost will be £$\frac{2478}{6}$, or £413.

The process may be written thus:

3*s*. 4*d*. is $\frac{1}{6}$ of £1 | £2478 = cost of the articles at £1 each.

£413 = cost .... at 3*s*. 4*d*. ...

Ex. (2). Find the cost of 2897 articles at £2. 12*s*. 9*d*. each.

£2 is 2 × £1	2897 .	0 .	0	= cost at £1 each.
10*s*. is $\frac{1}{2}$ of £1	5794 .	0 .	0	= ...... £2 ....
2*s*. is $\frac{1}{5}$ of 10*s*.	1448 .	10 .	0	= ...... 10*s*.....
8*d*. is $\frac{1}{3}$ of 2*s*.	289 .	14 .	0	= ...... 2*s*. ....
1*d*. is $\frac{1}{8}$ of 8*d*.	96 .	11 .	4	= ...... 8*d*. ....
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NOTE.—A shorter method would be to take the parts thus:

10*s*. = $\frac{1}{2}$ of £1; 2*s*. 6*d*. = $\frac{1}{4}$ of 10*s*.; 3*d*. = $\frac{1}{10}$ of 2*s*. 6*d*.

Ex. (3). Find the cost of 425 articles at £2. 18*s*. 4*d*. each.

Since £2. 18*s*. 4*d*. is the difference between £3 and 1*s*. 8*d*. (which is $\frac{1}{12}$ of £1), the shortest course is to find the cost at £3 each, and to *subtract from it* the cost at 1*s*. 8*d*. each, thus:

	£	*s*.	*d*.	
£3 is 3 × £1	425 .	0 .	0	= cost at £1 each.
1*s*. 8*d*. is $\frac{1}{12}$ of £1	1275 .	0 .	0	= ...... £3 ....
	35 .	8 .	4	= ...... 1*s*. 8*d*. each.
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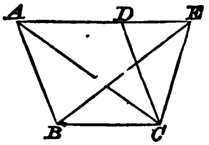
PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.



Let the ▱ *ABCD* and the △ *EBC* be on the same base *BC* and between the same ||s *AE*, *BC*.

Then must ▱ ABCD be double of △ EBC.

Join *AC*.

Then △ *ABC* = △ *EBC*, ∵ they are on the same base and between the same ||s ; I. 37.

and ▱ *ABCD* is double of △ *ABC*, ∵ *AC* is a diagonal of *ABCD* ; I. 34.

∴ ▱ *ABCD* is double of △ *EBC*.

Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

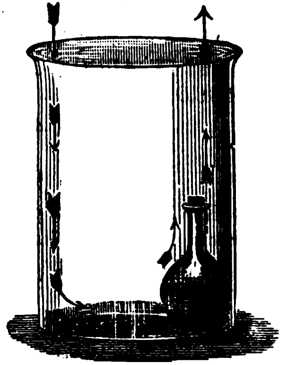
Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

Sometimes carbonic anhydride is produced in wells, and, being so much heavier than air, it remains at the bottom. If a man goes down into such a well, he will have no difficulty at first, because the air is good; but when he is near the bottom, where the gas has accumulated, he will gasp for breath and fall; and if anyone, not understanding the cause of his trouble, goes down to assist him, he too will fall senseless, and both will quickly die. The way to ascertain whether carbonic anhydride has accumulated at the bottom of a well is to let a light down into it. If it goes out, or even burns very dimly, there is enough of the gas to make the descent perilous. A man going down a well should always take a candle with him, which he should hold a considerable distance below his mouth. If the light burns dimly, he should at once stop, before his mouth gets any lower and he takes some of the gas into his lungs.

When this gas is in a well or pit, of course it must be expelled before a man can descend. There are several expedients for doing this. One is to let a bucket down frequently, turning it upside down, away from the mouth of the well, every time it is brought up, a plan which will remind you of the experiment represented in Fig. 24.

But a better way is to let down a bundle of burning straw or shavings, so as to heat the gas. Now heated bodies expand, gases very much more than solids or liquids, and, in expanding, the weight of a certain volume, say of a gallon, becomes lessened. So that if we can heat the carbonic anhydride enough to make a gallon of it weigh less than a gallon of air, it will rise out of the well just as hydrogen gas would do. Fig. 25 shows how you may perform this experiment upon a small scale.

Fig. 25.



DISASTROUS RETREAT OF THE ENGLISH FROM CABUL.

IT took two days of disorder, suffering, and death to carry the army, now an army no more, to the jaws of the fatal pass. Akbar Khan, who appeared like the Greeks' dread marshal from the spirit-land at intervals upon the route, here demanded four fresh hostages. The demand was acquiesced in. Madly along the narrow defile crowded the undistinguishable host, whose diminished numbers were still too numerous for speed: on every side rang the war-cry of the barbarians: on every side plundered and butchered the mountaineers: on every side, palsied with fatigue, terror, and cold, the soldiers dropped down to rise no more. The next day, in spite of all remonstrance, the general halted his army, expecting in vain provisions from Akbar Khan. That day the ladies, the children, and the married officers were given up. The march was resumed. By the following night not more than one-fourth of the original number survived. Even the haste which might once have saved now added nothing to the chances of life. In the middle of the pass a barrier was prepared. There twelve officers died sword in hand. A handful of the bravest or the strongest only reached the further side alive: as men hurry for life, they hurried on their way, but were surrounded and cut to pieces, all save a few that had yet escaped. Six officers better mounted or more fortunate than the rest, reached a spot within sixteen miles of the goal; but into the town itself rode painfully on a jaded steed, with the stump of a broken sword in his hand, but one.

LIVY, xxi. c. 25, § 7-10. xxxv. c. 30. xxiii. c. 24.
CÆSAR, *Bell. Gall.* v. c. 35-37.

DEFEAT OF CHARLES THE BOLD AND MASSACRE OF HIS TROOPS AT MORAT.

IN such a predicament braver soldiers might well have ceased to struggle. The poor wretches, Italians and Savoyards, six thousand or more in number, threw away their arms and made

[*Specimen Page, No.* 11.]

II.

ARIADNE'S LAMENT.

> **Madam, 'twas Ariadne passioning**
> **For Theseus' perjury and unjust flight.**
> **TWO GENTLEMEN OF VERONA, IV. 4, 172.**

ARGUMENT.

ARIADNE *tells the story of her first waking, to find herself abandoned by Theseus and left on an unknown island, exposed to a host of dangers.*—(HEROIDES, x.)

The story is beautifully told by Catullus, in the "Epithalamium Pelei et Thetidos:" it also forms one of the episodes in Chaucer's "Legende of Goode Women."

I woke before it was day to find myself alone, no trace of my companions to be seen. In vain I felt and called for Theseus; the echoes alone gave me answer.

QUAE legis, ex illo, Theseu, tibi litore mitto,
  Unde tuam sine me vela tulere ratem:
In quo me somnusque meus male prodidit et tu,
  Per facinus somnis insidiate meis. **107**
Tempus erat, vitrea quo primum terra pruina **112**
  Spargitur et tectae fronde queruntur aves:
Incertum vigilans, a somno languida, movi **97**
  Thesea prensuras semisupina manus:
Nullus erat, referoque manus, iterumque retempto,
  Perque torum moveo brachia: nullus erat.
Excussere metus somnum: conterrita surgo,
  Membraque sunt viduo praecipitata toro. **123**
Protinus adductis sonuerunt pectora palmis, **111**
  Utque erat e somno turbida, rapta coma est.
Luna fuit: specto, siquid nisi litora cernam;
  Quod videant, oculi nil nisi litus habent. **150**
Nunc huc, nunc illuc, et utroque sine ordine curro;
  Alta puellares tardat arena pedes.
Interea toto clamanti litore "Theseu!" **121**
  Reddebant nomen concava saxa tuum,
Et quoties ego te, toties locus ipse vocabat:
  Ipse locus miserae ferre volebat opem. **106** 3

[TAYLOR'S OVID—*See Page* 16.]

[*Specimen Page, No.* 12.]

STORIES FROM OVID.

174. **Punica poma,** pomegranates.

178. **Taenarum,** at the southern extremity of Peloponnesus, was one of the numerous descents to Tartarus. Cf. Virgil, Georg. IV. 467:

> Taenarias etiam fauces, alta ostia Ditis.

179. **Factura fuit.** This periphrasis for *fecisset* is to be noted; it is the one from which the oblique forms are all constructed, *e.g.*, *facturam fuisse*, or *factura fuisset.*

183. **Cessatis,** one of a goodly number of intransitive verbs of the first conjugation which have a passive participle. Cf. **erratas,** above, 139, **clamata,** 35. So Horace, regnata Phalanto rura (Odes, II. 6, 12); triumphatae gentes (Virgil).

II.—IV.

ARIADNE.

THIS and the two following extracts, though taken from different works, form a definite sequence. Ariadne, daughter of Minos, king of Crete, has helped Theseus to conquer the Minotaur, by giving him a clew to the maze in which the monster was hid, and, being in love with him, has fled in his company. They put in for the night to the island of Dia, and Theseus on the next morning treacherously sails away, leaving the poor girl alone. The first extract is part of an epistle which she is supposed to write on the day when she discovers his perfidy.

The name Dia, which belonged properly to a small island off the north coast of Crete, was also a poetical name for Naxos, one of the largest of the Cyclades. It may have been this fact which led to the further legend which is recounted in the next extract, how Ariadne, lorn of Theseus, becomes the bride of Bacchus; for Naxos was the home of the Bacchic worship. As the completion of the legend she is raised to share in Bacchus' divine honours, and as the Cretan Crown becomes one of the signs of the heavens.

II.

ARIADNE'S LAMENT.

1. **Illo,** sc. *Diae.*

4. **Per facinus,** criminally.

5. Describing apparently the early dawn, or the hour that precedes it, when the night is at its coldest, and the birds, half-awake, begin to stir in their nests. **Pruina** hints that it is autumn.

7. A beautifully descriptive line—But half-awake, with all the languor of sleep still on me.
A somno = after, as the *result* of.

8. **Semisupina,** on my side, lit., half on my back, describes the motion of a person thus groping about on waking. Cf. Chaucer:

> Ryght in the dawenynge awaketh shee,
> And gropeth in the bed, and fonde ryghte noghte.

[TAYLOR'S OVID—*See Page* 16.]

haec mea magna fides? at non, Euandre, pudendis
volneribus pulsum aspicies, nec sospite dirum
optabis nato funus pater. ei mihi, quantum
praesidium Ausonia, et quantum tu perdis, Iule!
    Haec ubi deflevit, tolli miserabile corpus
imperat, et toto lectos ex agmine mittit
mille viros, qui supremum comitentur honorem,
intersintque patris lacrimis, solacia luctus
exigua ingentis, misero set debita patri.
haut segnes alii crates et molle feretrum
arbuteis texunt virgis et vimine querno,
extructosque toros obtentu frondis inumbrant.
hic iuvenem agresti sublimem stramine ponunt;
qualem virgineo demessum pollice florem
seu mollis violae, seu languentis hyacinthi,
cui neque fulgor adhuc, nec dum sua forma recessit;
non iam mater alit tellus, viresque ministrat.
tunc geminas vestes auroque ostroque rigentis
extulit Aeneas, quas illi laeta laborum
ipsa suis quondam manibus Sidonia Dido
fecerat, et tenui telas discreverat auro.
harum unam iuveni supremum maestus honorem
induit, arsurasque comas obnubit amictu;
multaque praeterea Laurentis praemia pugnae
aggerat, et longo praedam iubet ordine duci.
addit equos et tela, quibus spoliaverat hostem.
vinxerat et post terga manus, quos mitteret umbris
inferias, caeso sparsuros sanguine flammam;
indutosque iubet truncos hostilibus armis
ipsos ferre duces, inimicaque nomina figi.
ducitur infelix aevo confectus Acoetes,
pectora nunc foedans pugnis, nunc unguibus ora;
sternitur et toto proiectus corpore terrae.

[STORR'S VERGIL—*See Page* 16.]

Comp. *Geor.* ii. 80, *Nec longum tempus et . . . exiit . . . arbos*, C. But as these are the only two instances of the construction adduced it is perhaps safer to take *et* = even.

51 **nil iam**, etc.] The father is making vows to heaven in his son's behalf, but the son is gone where vows are neither made nor paid.

55 **haec mea magna fides**] 'Is this the end of all my promises?' *Magna* may be taken as 'solemn,' or 'boastful.'

pudendis volneribus] All his wounds are on his breast.

56 **dirum optabis funus** = *morti devovebis*. Compare the meaning of *dirae*, xii. 845.

59-99] A description of the funeral rites. Aeneas bids his last farewell.

59 **Haec ubi deflevit**] 'His moan thus made.' *De* in composition has two opposite meanings: (1) cessation from or removal of the fundamental ideas, as in *decresco*, *dedoceo*, etc.; (2) (as here) in intensifying, as *debello*, *demiror*, *desaevio*.

61 **honorem**] *Honos* is used by V. for (1) a sacrifice, iii. 118; (2) a hymn, *Geor.* ii. 393; (3) beauty, *Aen.* x. 24; (4) the 'leafy honours' of trees, *Geor.* ii. 404; (5) funeral rites, vi. 333, and here. See below, *l.* 76.

63 **solatia**] In apposition to the whole sentence; whether it is nom. or acc. depends on how we resolve the principal sentence; here, though *solatia* applies to the whole sentence, its construction probably depends on the last clause, which we may paraphrase, *ut praesentes* (τὸ μετεῖναι) *sint solatia;* therefore it is nom.

64 **crates et molle feretrum**] The bier of pliant osier: cf. *l.* 22.

66] Cf. Statius, *Theb.* vi. 55, *torus* et puerile feretrum.

obtentu frondis] 'A leafy canopy.' C. understands '**a layer of leaves.**

67 **agresti stramine**] 'The rude litter.'

68] Cf. ix. 435; *Il.* viii. 306,

> *μήκων δ᾽ ὡς ἑτέρωσε κάρη βάλεν, ἥτ᾽ ἐνὶ κήπῳ*
> *καρπῷ βριθομένη νοτίῃσί τε εἰαρινῇσιν·*
> *ὣς ἑτέρωσ᾽ ἤμυσε κάρη πήληκι βαρυνθέν.*

> 'Even as a flower,
> Poppy or hyacinth, on its broken stem
> Languidly raises its encumbered head.'—MILMAN.

69 **languentis hyacinthi**] The rhythm is Greek. The 'drooping hyacinth' is probably the Lilium Martagon or Turk's-cap lily, 'the sanguine flower inscribed with woe.'

70] 'That hath not yet lost its gloss nor all its native loveliness.' *Recessit* must apply to both clauses. 'If we suppose the two parts of the line to contain a contrast, the following line will lose much of its force,' C. Compare the well-known lines from the *Giaour*, 'He who hath bent him o'er the dead,' etc.

71] Contrast the force of *neque adhuc*, *nec dum*, and *non iam;* 'the brightness not all gone,' 'the lines where beauty lingers,' and 'the support and nurture of mother earth cut off once and for all.'

36. *ἵνα φάγῃ*] In modern Greek, which properly speaking has no infinitive, the sense of the infinitive is expressed by *νά* (*ἵνα*) with subjunctive (as in this passage), *e.g.* *ἐπιθυμῶ νὰ νράφῃ*, 'I wish him to write;' see Corfe's *Modern Greek Grammar*, p. 78. This extension of the force of *ἵνα* to oblique petition, and even to consecutive clauses, may be partly due to the influence of the Latin *ut;* cf. ch. xvi. 27, *ἐρωτῶ οὖν, πάτερ, ἵνα πέμψῃς*: see note on ch. iv. 3.

The following incident is recorded by St. Luke alone. Simon the Pharisee is not to be identified with Simon the leper, Matt. xxvi., Mark xiv. 3.

ἀνεκλίθη] The Jews had adopted the Roman, or rather Greek, fashion of reclining at meals—a sign of advancing luxury and of Hellenism, in which however even the Pharisee acquiesces.

37. *γυνή*] There is no proof that this woman was Mary Magdalene. But mediæval art has identified the two, and great pictures have almost disarmed argument in this as in other incidents of the gospel narrative.

38. *ἀλάβαστρον*] The neuter sing. is Hellenistic. The classical form is *ἀλάβαστρος* with a heteroclite plural *ἀλάβαστρα*, hence probably the late sing. *ἀλάβαστρον*. The grammarian stage of a language loves uniformity, Herod. iii. 20; Theocr. xv. 114:

> *Συρίω δὲ μύρω χρύσει' ἀλάβαστρα.*

στᾶσα παρὰ τοὺς πόδας αὐτοῦ] This would be possible from the arrangement of the triclinium.

39. *ἐγίνωσκεν ἂν*] 'Would (all the while) have been recognising.'

40. *χρεωφειλέται*] A late word; the form varies between *χρεωφειλέται* and *χρεοφειλέται*.

41. *δηνάρια*] The denarius was a silver coin originally containing ten ases (deni), afterwards, when the weight of the as was reduced, sixteen ases. Its equivalent modern value is reckoned at 7½d. But such calculations are misleading; it is more to the point to regard the denarius as an average day's pay for a labourer.

42. *μὴ ἐχόντων*] Because *he saw that* they had not.

ἐχαρίσατο] Cf. *v.* 21.

[CARR'S NOTES ON THE GREEK TESTAMENT.—*See Page* 23.]

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